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THE EQUIVALENCE OF TWO DEFINITIONS OF CAPACITY[1]

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ABSTRACT. It is shown that two definitions for an L_p capacity ($1<p<\infty$) on subsets of Euclidean $\boldsymbol{R}^n$ are equivalent in the sense that as set functions their ratio is bounded above and below by positive finite constants. The classical notions of capacity correspond to the case $p=2$.

1. Let $L_{\alpha,p}=g_\alpha(L_p)$, $1<p<\infty$, $\alpha>0$, where g_α is the L_1^+ function on $\boldsymbol{R}^n$ which is the Fourier transform of $(2\pi)^{-n/2}(1+|\xi|^2)^{-\alpha/2}$, $\xi\in\boldsymbol{R}^n$. $L_p=L_p(\boldsymbol{R}^n)$ are the usual Lebesgue spaces.

DEFINITION 1. $B_{\alpha,p}(A)=\inf \|f\|_p^p$, where the infimum is over all $f\in L_p^+$ for which $g_\alpha * f(x)\geqq 1$ on A, $A\subset\boldsymbol{R}^n$.

DEFINITION 2. $C_{\alpha,p}(K)=\inf \|\varphi\|_{\alpha,p}^p$, where the infimum is over all $\varphi\in C_0^\infty(\boldsymbol{R}^n)$ for which $\varphi(x)\equiv 1$ in a neighborhood of K, K compact set in $\boldsymbol{R}^n$.

Here $\|\cdot\|_{\alpha,p}$ denotes the usual norm in $L_{\alpha,p}$ and $\|\cdot\|_{0,p}=\|\cdot\|_p$. The purpose of this paper is to show

THEOREM A. *For all compact sets $K\subset\boldsymbol{R}^n$,*

$$B_{\alpha,p}(K)\sim C_{\alpha,p}(K).$$

Here $\sim$ means that the ratio is bounded above and below by positive finite constants independent of the set K.

REMARK. The set function $C_{\alpha,p}$ is extended to the class of all subsets by $C_{\alpha,p}(A)=\sup C_{\alpha,p}(K)$, where the supremum is over all compact sets K contained in A. When this is done, the equivalence of the theorem extends to all capacitable sets and in particular to all analytic sets. For details see [13].

As an example of the utility of these capacities, we state the following removable singularity theorem. Let $P(x, D)$ be a partial differential operator of order m defined in an open set $\Omega\subset\boldsymbol{R}^n$.

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THEOREM B. *Let* $1<p<\infty$ *and* $1/p+1/q=1$. *Let* $A\subset\Omega$ *be relatively closed.*

(a) *If* $B_{m,q}(A)=0$, *then each* f *which is locally in* $L_p(\Omega)$ *and satisfies* $P(x, D)f=0$ *in* $\Omega-A$, *also satisfies* $P(x, D)f=0$ *in* Ω.

(b) *Suppose* $P(x, D)$ *is elliptic and has a fundamental solution in* Ω. *If each* f, *locally in* $L_p(\Omega)$, *which satisfies* $P(x, D)f=0$ *in* $\Omega-A$ *also satisfies* $P(x, D)f=0$ *in* Ω, *then* $B_{m,q}(A)=0$.

Theorem B is an easy corollary of Theorem A and the work of Littman [**8**], [**9**]. (See also [**6**].) As far as the authors know, Theorem B has not appeared explicitly in the literature though it seems to be widely known. For the case where $P(x, D)$ is a power of the Laplacian operator, Theorem B is implicit in the work of Maz'ja and Havin [**10**]. Their removable singularity results are stated in terms of Hausdorff measure instead of capacity. These results and the L^p removable singularity results in [**5**] follow directly from Theorem B and the relationship between $B_{\alpha,p}$ and Hausdorff measure (see [**10**], [**13**], or [**16**]).

The capacities defined above have been the subject of much investigation of late. $B_{\alpha,p}$ and closely related capacities have been studied by Fuglede [**3**], [**4**], Meyers [**13**], Adams and Meyers [**1**], and Rešetnjak [**16**], Maz'ja and Havin [**10**]. $C_{\alpha,p}$ has been studied by Littman [**8**], Maz'ja [**12**], and Harvey and Polking [**6**]. Maz'ja [**11**] has announced Theorem A in the case of integral α.

To prove Theorem A, we first notice that $B_{\alpha,p}(K)\leqq C_{\alpha,p}(K)$. This fact is elementary and only uses the fact that the kernels g_α are nonnegative. If $\varphi\in C_0^\infty(\boldsymbol{R}^n)$ and $\varphi(x)=1$ in a neighborhood of K, then $\varphi=g_\alpha*\psi$ for some $\psi\in L_p$. Let $\psi_+(x)=\max(0, \psi(x))$. Then $\psi_+\in L_p^+$, $\|\psi_+\|_p\leqq\|\psi\|_p=\|\varphi\|_{\alpha,p}$, and $g_\alpha*\psi_+(x)\geqq g_\alpha*\psi(x)=1$ for all $x\in K$.

To derive the reverse inequality, we use a method due to Littman [**8**]. The key fact that makes the method work is the boundedness principle for capacity extremals. The form that we will use is the following: Let G be an open set in $\boldsymbol{R}^n$. Then there is a function $f\in L_p^+$ such that: (a) $g_\alpha*f(x)\geqq 1$ on G, (b) $g_\alpha*f(x)\leqq Q<\infty$ for all $x\in\boldsymbol{R}^n$, where Q is a constant depending only on n and p, and (c) $\|f\|_p^p=B_{\alpha,p}(G)$. (This result was obtained independently by Adams and Meyers [**1**] and Maz'ja and Havin [**10**].) So, for K compact we choose such a $G\supset K$. Furthermore, we can find G_1 open: $K\subset G_1\subset G$ and a C_0^∞ regularization φ of $g_\alpha*f$ (f chosen as above for G) such that: (a′) $\varphi(x)\geqq 1$ on G_1, (b′) $0\leqq\varphi(x)\leqq Q$, for all x (same Q), and (c′) $\|\varphi\|_{\alpha,p}\leqq\text{const }\|f\|_p$. Now let $H(t)$, $t\geqq 0$, be a $C^\infty(\boldsymbol{R}^1_+)$ function such that $H(t)=t$, $0\leqq t\leqq\frac{1}{2}$, and $H(t)=1$, $t\geqq 1$. Clearly, $\|H(\varphi)\|_p\leqq\text{const }\|\varphi\|_p$. The proof of the theorem will now be concluded by showing: for any $\alpha>0$,

$$\|H(\varphi)\|_{\alpha,p}\leqq Q_1\|\varphi\|_{\alpha,p}\left(\sum_{j<\alpha}\|\varphi\|_\infty^j\right) \tag{1}$$

for some constant Q_1 independent of φ. To see that (1) completes the proof, we need only observe that $H(\varphi)\equiv 1$ on G_1, $H(\varphi)\in C_0^\infty(\boldsymbol{R}^n)$ and

$$C_{\alpha,p}(K) \leqq \|H(\varphi)\|_{\alpha,p}^p \leqq \text{const } \|f\|_p^p = \text{const } B_{\alpha,p}(G)$$

by the known properties of φ. The desired inequality then follows by the capacitability results of **[13]**. Note the above const is independent of the sets K and G.

PROOF OF (1). We consider two cases.

Case 1. α=integer. We need some preliminary theorems.

THEOREM 1 (**[2]**). *For $1<p<\infty$ and α integral,*

$$\|\varphi\|_{\alpha,p} \sim \sum_{0\leqq|s|\leqq\alpha} \|D^s\varphi\|_p.$$

D^s denotes derivatives in the coordinate directions $s=(s_1,\cdots,s_n)$, $|s|=s_1+\cdots+s_n$.

THEOREM 2. *For $0<\lambda<1$, $0<\alpha<\infty$ (α not necessarily integral),*

$$\|\varphi\|_{\alpha(1-\lambda),p/(1-\lambda)} \leqq c\, \|\varphi\|_{\alpha,p}^{1-\lambda}\, \|\varphi\|_\infty^\lambda,$$

c independent of φ.

For periodic functions on $[0, 2\pi]$, this theorem is due to Hirschman **[7]**. For integral α and $\alpha(1-\lambda)$, it is the well-known inequality of Nirenberg and Gagliardo (see **[14]**). A proof of Theorem 2 may be found in **[1]**.

Since for real β, γ, $0\leqq\beta\leqq\gamma$, $\|u\|_{\beta,p}\leqq\|u\|_{\gamma,p}$, we need only concentrate on estimating $D^sH(\varphi)$ in L_p, $|s|=\alpha$. There are positive constants $C_{r^1,\dots,r^j}$ depending only on the multi-index subscripts such that

$$D^sH(\varphi) = \sum_{j=1}^{\alpha} H^{(j)}(\varphi) \sum C_{r^1,\dots,r^j} D^{r^1}\varphi\cdots D^{r^j}\varphi, \tag{2}$$

with the last sum over all sets of multi-indices $(r^1,\cdots,r^j)$, such that $|r^i|\geqq 1$ and $r^1+\cdots+r^j=s$. Hölder's inequality gives

$$\left\|\prod_{l=1}^{j} D^{r^l}\varphi\right\|_p \leqq \prod_{l=1}^{j} \|D^{r^l}\varphi\|_{pp_l},$$

$\sum_l (1/p_l)=1$, $1<p_l<\infty$. Choose $p_l=\alpha/|r^l|$. Then

$$\|D^{r^l}\varphi\|_{pp_l} \leqq c\, \|\varphi\|_{\alpha/p_l,pp_l} \leqq c\, \|\varphi\|_{\alpha,p}^{1/p_l}\, \|\varphi\|_\infty^{1-1/p_l}.$$

Case 2. $\alpha \neq$integer. The preliminary results needed here are

THEOREM 3. $\varphi \in L_{\alpha,p}$, $\alpha > 1$, *if and only if* $D^s\varphi \in L_{\alpha-|s|,p}$, $0 \leqq |s| \leqq [\alpha]$, *and*

$$\|\varphi\|_{\alpha,p} \sim \sum_{0\leqq |s| \leqq [\alpha]} \|D^s\varphi\|_{\alpha-[\alpha],p}.$$

Here $[\alpha]$=*greatest integer in* α.

Theorem 3 is an easy corollary of Theorem 1 and the definition of the space $L_{\alpha,p}$.

Let

$$\mathscr{D}_p^\alpha(u)(x) = \left(\int_0^\infty \left(\int_{|y|\leqq 1} |u(x+\rho y) - u(x)|^p \, dy\right)^2 \frac{d\rho}{\rho^{1+2\alpha p}}\right)^{1/2p}.$$

Then

THEOREM 4. (a) ([15]). *If* $1 \leqq p < \infty$, $0 < \alpha < 1$, *and* $\max(1, np/(n+\alpha p)) < r < \infty$, *then* $\|\mathscr{D}_p^\alpha(u)\|_r \leqq c\|u\|_{\alpha,r}$, *all* $u \in L_{\alpha,r}$.

(b) ([17]) $\|\mathscr{D}_1^\alpha(u)\|_r + \|u\|_r \sim \|u\|_{\alpha,r}$, $1 < r < \infty$, $0 < \alpha < 1$.

We begin with $0 < \alpha < 1$.

$$\begin{aligned} \|H(\varphi)\|_{\alpha,p} &\leqq c\, \|\mathscr{D}_1^\alpha H(\varphi)\|_p + c\, \|H(\varphi)\|_p \\ &\leqq c\, \|\mathscr{D}_1^\alpha(\varphi)\|_p + c\, \|\varphi\|_p \leqq c\, \|\varphi\|_{\alpha,p}. \end{aligned}$$

Note $\mathscr{D}_1^\alpha H(\varphi) \leqq c\mathscr{D}_1^\alpha(\varphi)$ since H is Lipschitz continuous.

Now assume $\alpha = k+\sigma$, k a positive integer and $0 < \sigma < 1$.

$$\|H(\varphi)\|_{\alpha,p} \leqq c \sum_{0\leqq |s| \leqq k} \|D^s H(\varphi)\|_{\sigma,p}.$$

Again we need only consider the case $|s| = k$.

$$\|D^s H(\varphi)\|_{\sigma,p} \leqq c\, \|\mathscr{D}_1^\sigma(D^s H(\varphi))\|_p + c\, \|D^s H(\varphi)\|_p.$$

The last term has already been treated. Now using (2) with α replaced by k, it easily follows that

$$\mathscr{D}_1^\sigma(D^s H(\varphi)) \leqq c\mathscr{D}_1^\sigma(D^{r^1}\varphi \cdots D^{r^j}\varphi) + c\, |D^{r^1}\varphi \cdots D^{r^j}\varphi|\, \mathscr{D}_1^\sigma(\varphi),$$

again noting $\mathscr{D}_1^\sigma(H^{(j)}(\varphi)) \leqq c\mathscr{D}_1^\sigma(\varphi)$. The L_p norm of the last term in the above sum does not exceed $c\|D^{r^1}\varphi\|_{\alpha p/|r^1|} \cdots \|D^{r^j}\varphi\|_{\alpha p/|r^j|}\|\mathscr{D}_1^\sigma(\varphi)\|_{\alpha p/\sigma} \leqq c\|\varphi\|_{\alpha,p}\|\varphi\|_\infty^j$, upon applying Hölder's inequality and Theorems 4, 1 and 2.

To treat the remaining terms, we use

THEOREM 5 ([15]). *If* $0 < \lambda$, $\mu < 1$, *then*

$$\mathscr{D}^\sigma(fg) \leqq |f|\, \mathscr{D}_p^\sigma(g) + |g|\, \mathscr{D}_p^\sigma(f) + \mathscr{D}_{p/\mu}^{\lambda\sigma}(f)\mathscr{D}_{p/(1-\mu)}^{(1-\lambda)\sigma}(g).$$

Using the convention $\mathscr{D}^0_\infty(f)=|f|$, repeated applications of Theorem 5 to $\mathscr{D}^\sigma_1(D^{r^1}\varphi\cdots D^{r^j}\varphi)$ show that this expression is bounded by a finite sum of terms of the form

$$\mathscr{D}^{\sigma_1}_{q_1}(D^{r^1}\varphi)\cdots\mathscr{D}^{\sigma_j}_{q_j}(D^{r^j}\varphi). \tag{3}$$

The number of terms depends only on j. Here $\sigma_i\geqq 0$ and $q_i\geqq 1$ ($q_k=\infty$ when and only when $\sigma_i=0$), and $\sum\sigma_i=\sigma$, $\sum(1/q_i)=1$. In addition, Theorem 5 allows us considerable freedom in choosing the nonzero σ_i's and finite q_i's; this permits us to make a particular choice so that Theorem 4(a) applies below. Namely, we choose $q_i\leqq\alpha/(\sigma_i+|r^i|)$ when $q_i<\infty$. Now setting $p_i=\alpha p/(\sigma_i+|r^i|)$, we have, applying Hölder's inequality together with Theorems 4, 3 and 2, that the L_p norm of (3) does not exceed

$$\prod_{i=1}^{j}\|\mathscr{D}^{\sigma_i}_{q_i}(D^{r^i}\varphi)\|_{p_i}\leqq c\prod\|D^{r^i}\varphi\|_{\sigma_i,p_i}\leqq c\prod\|\varphi\|_{\sigma_i+|r^i|,p_i}$$
$$\leqq c\,\|\varphi\|_{\alpha,p}\,\|\varphi\|_\infty^{j-1}.$$

REFERENCES

1. D. R. Adams and N. G. Meyers, *Bessel potentials. Inclusion relations among classes of exceptional sets*, Indiana Univ. Math. J. (to appear). (An announcement of these results appears in Bull. Amer. Math. Soc. **77** (1971), 968–970.)

2. A. P. Calderón, *Lebesgue spaces of differentiable functions and distributions*, Proc. Sympos. Pure Math., vol. 4, Amer. Math. Soc., Providence, R.I., 1961, pp. 33–49. MR **26** #603.

3. B. Fuglede, *Extremal length and functional completion*, Acta Math. **98** (1957), 171–219. MR **20** #4187.

4. ———, *Application du théorème minimax a l'étude diverse capacités*, C. R. Acad. Sci. Paris **266** (1968), 921–923.

5. R. Harvey and J. C. Polking, *Removable singularities of solutions of linear partial differential equations*, Acta Math. **125** (1970), 39–56. MR **43** #5183.

6. ———, *A notion of capacity which characterizes removable singularities*, Trans. Amer. Math. Soc. **169** (1972), 183–195.

7. I. I. Hirschman, *A convexity theorem for certain groups of transformations*, J. Analyse Math. **2** (1952/53), 209–218. MR **15**, 295; 1139.

8. W. Littman, *A connection between α-capacity and m-p polarity*, Bull. Amer. Math. Soc. **73** (1967), 862–866. MR **36** #2940.

9. ———, *Polar sets and removable singularities of partial differential equations*, Ark. Mat. **7** (1967), 1–9. MR **37** #559.

10. V. G. Maz'ja and V. P. Havin, *A nonlinear analogue of the Newtonian potential and metric properties of (p, 1)-capacity*, Dokl. Akad. Nauk SSSR **194** (1970), 770–773=Soviet Math. Dokl. **11** (1970), 1294–1298. MR **42** #7926.

11. V. G. Maz'ja, *Imbedding theorems and their applications*, Baku Sympos. (1966), "Nauka", Moscow, 1970, pp. 142–159. (Russian).

12. ———, *p-conductance and theorems of imbedding certain function spaces into the space* $\mathfrak{C}$, Dokl. Akad. Nauk SSSR **140** (1961), 299–302=Soviet Math. Dokl. **2** (1961), 1200–1203. MR **28** #460.

13. N. G. Meyers, *A theory of capacities for potentials of functions in Lebesgue classes*, Math. Scand. **26** (1970), 255–292. MR **43** #3474.

14. L. Nirenberg, *On elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa (3) **13** (1959), 115–162. MR **22** #823.

15. J. C. Polking, *A Leibniz formula for some differentiation operators of fractional order*, Indiana Univ. Math. J. **21** (1972), 1019–1029.

16. Ju. G. Rešetnjak, *The concept of capacity in the theory of functions with generalized derivatives*, Sibirsk. Mat. Ž. **10** (1969), 1109–1138=Siberian Math. J. **10** (1969), 818–842. MR **43** #2234.

17. R. S. Strichartz, *Multipliers on fractional Sobolev spaces*, J. Math. Mech. **16** (1967), 1031–1060. MR **35** #5927.

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