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3-MANIFOLDS THAT ARE SUMS OF SOLID TORI AND SEIFERT FIBER SPACES

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ABSTRACT. It is shown that a simply connected 3-manifold is S^3 if it is a sum of a Seifert fiber space and solid tori. Let F be an orientable Seifert fiber space with a disk as orbit surface. It is shown that a sum of F and a solid torus is a Seifert fiber space or a connected sum of lens spaces.

Let M be a closed 3-manifold which is a union of three solid tori. It is shown that M is a Seifert fiber space or the connected sum of two lens spaces (including $S^1 \times S^2$).

Let M be a sum of a Seifert fiber space and solid tori. It is shown (Theorem 1) that if M is simply connected, then M is S^3. This generalizes Hempel's result in [4]. Let F be a Seifert fiber space with orbit surface a disk. A particular example is the complement (of a regular neighborhood in S^3) of a torus knot. The structure of all 3 manifolds N that are a sum of F and a solid torus is described (Theorem 3). Also the structure of all 3-manifolds that are a union of three solid tori (such that the intersection of any two is an annulus) is described (Theorem 4). In particular a question of A. C. Connor [3] is answered in the affirmative, that any such 3-manifold is S^3 if it is simply connected.

1. We work throughout in the piecewise linear category. A *sum* of a 3-manifold F and solid tori $V_1, \cdots, V_n$ is the manifold obtained from F and $V_1, \cdots, V_n$ by identifying components T_i of ∂F with ∂V_i under homeomorphisms $f_i: \partial V_i \to T_i$ $(i=1, \cdots, n)$. The *connected sum* $M_1 \# M_2$ of two 3-manifolds is the manifold obtained by removing 3-balls in $\operatorname{int}(M_1)$ and $\operatorname{int}(M_2)$, and identifying the resulting 2-sphere boundaries under an orientation reversing homeomorphism. For a definition and classification of Seifert fiber spaces, see [5] and [6]. If F is a Seifert fiber space there is a map P of F onto its orbit surface f (Zerlegungsfläche). The image of an exceptional fiber is an exceptional point on f.

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2. **Simply connected 3-manifolds.** Let m_i be a meridian curve on ∂V_i ($m_i \sim 0$ in V_i but $m_i \nsim 0$ on ∂V_i).

LEMMA 1. *Suppose M is a sum of a Seifert fiber space F and solid tori V_i ($i=1, \cdots, r$; $r \geqq 2$). If $f_1(m_1)$ and $f_2(m_2)$ are homologous (on ∂V_1 and ∂V_2 respectively) to a fiber, then M contains a nonseparating 2-sphere.*

PROOF. We can assume that $f_i(m_i)$ is a fiber, $i=1, 2$. Let P be the projection of F onto the orbit surface f (Zerlegungsfläche). Choose a (nonseparating) arc c on f which does not meet exceptional points and which joins $Pf_1(m_1)$ and $Pf_2(m_2)$. Then $P^{-1}(c)$ is an annulus A with boundary $f_1(m_1)$ and $f_2(m_2)$ which bound disks D_1 and D_2 in $\mathrm{Cl}(M-F)$. Thus $D_1 \cup A \cup D_2$ is a nonseparating 2-sphere.

LEMMA 2. *Suppose M is a simply connected manifold which is a sum of a Seifert fiber space F and a solid torus V such that $f(m) \sim$ fiber (on ∂F) then F is a solid torus (and without exceptional fibers).*

PROOF. Clearly $\partial F = \partial V$ and F is orientable. We obtain a presentation for $\pi_1(F)$ as follows (see [7]). Let $D_1, \cdots, D_r$ be disjoint disks in the orbit surface f such that D_i contains the ith exceptional fiber in its interior and such that $D_i \cap F$ is an arc in ∂f ($i=1, \cdots, r$). Then $F' = P^{-1}(\mathrm{Cl}(f - \bigcup D_j))$ is a fiberbundle with fiber S^1. Therefore $\pi_1(F') = \{x_1, \cdots, x_k, h : x_i^{-1} h x_i = h^{\varepsilon_i}, \varepsilon_i = \pm 1\}$ where $k=2g$ if f is orientable of genus g and $k=g$ if f is nonorientable of genus g. Now $V_j = P^{-1}(D_j)$ is a solid torus, the image of $\pi_1(F' \cap V_j)$ in $\pi_1(F')$ is $\{h\}$ and the image in $\pi_1(V_j)$ has index μ_i, where $|\mu_i| \geqq 2$ (the order of the ith exceptional fiber). Therefore

$$\pi_1(F) = \{x_1, \cdots, x_k, y_1, \cdots, y_r, h : x_i^{-1} h x_i = h^{\varepsilon_i}, y_j^{\mu_j} = h\}.$$

$\pi_1(M)$ is obtained from $\pi_1(F)$ by adding the relation $h=1$, and is a free product of $k+r$ nontrivial cyclic factors. Therefore $\pi_1(M)=1$ implies that $k=r=0$, which means that the orbit surface f is a disk without exceptional points and $F=P^{-1}(f)$ is a fibered solid torus without exceptional fibers.

THEOREM 1. *A simply connected 3-manifold is S^3 if it is a sum of a Seifert fiber space F and solid tori.*

PROOF. Since ∂F has no 2-sphere components and since $\pi_1(M)=1$, it follows that M is closed. Therefore ∂F consists of tori T_i which are to be identified with the boundary of solid tori V_i ($i=1, \cdots, n$). Let m_i be a meridian curve on ∂V_i. If, for each $i=1, \cdots, n$, $f_i(m_i)$ is not homologous to a fiber on T_i, then the fibering on T_i can be extended to a fibering on V_i. Then M is a Seifert fiber space. By [5, Satz 11], M is S^3. If $f_i(m_i)$ is homologous to a fiber for at least two i, then M cannot be simply connected by

Lemma 1. Therefore assume $f_1(m_1) \sim$ fiber, but $f_i(m_i) \not\sim$ fiber for $i=2, \cdots, n$. Let M' be the sum of F and the solid tori V_i $(i=2, \cdots, n)$. Then M' is a Seifert fiber space and therefore a solid torus by Lemma 2. (In fact the proof of Lemma 2 shows that F is already a solid torus.) It follows that $M=M' \cup V_1$ is S^3.

COROLLARY [4]. *A simply connected 3-manifold is S^3 if it is a sum of a solid torus and the complement of a torus knot.*

COROLLARY 1. *A simply connected 3-manifold is S^3 if it is a sum of solid tori and the complement of a link in S^3 whose group has nontrivial center.*

PROOF. It follows from [7] that the complement of such a link is a Seifert fiber space. For a classification of these links see [1].

THEOREM 2. *A simply connected 3-manifold is S^3 if it is a union of three solid tori (such that the intersection of any two of them is an annulus).*

PROOF. Let $M=V_1 \cup V_2 \cup V_3$, a sum of three solid tori and let $V_i \cap V_j = A_{ij}$ $(i, j=1, 2, 3, i \neq j)$ an annulus. Let a be a component of $A_{12} \cap A_{13} \cap A_{23}$ and let m_i on ∂V_i be a meridian of V_i $(i=1, 2, 3)$.

If $a \not\sim m_1$ and $a \not\sim m_2$ on ∂V_1 and ∂V_2, respectively, we can fiber V_1 and V_2 (in the sense of Seifert) such that a is a (ordinary) fiber. Then $N=V_1 \cup V_2$ is a Seifert fiber space and M is a sum of N and the solid torus V_3. By Theorem 1, if M is simply connected, then M is S^3. The same argument applies if $a \not\sim m_1$, $a \not\sim m_3$ and if $a \not\sim m_2$, $a \not\sim m_3$.

Therefore we can assume that $a \sim m_1$ on ∂V_1 and $a \sim m_2$ on ∂V_2. But then $\pi_1(N)=\boldsymbol{Z} * \boldsymbol{Z}$ and $\pi_1(M)$ is obtained from $\pi_1(N)$ by adding one relation (which describes the curve on ∂N that is identified with a meridian m_3 on ∂V_3). Therefore $\pi_1(M)$ cannot be trivial, a contradiction.

3. **Sums of some Seifert fiber spaces and solid tori.** A lens space is a sum of two solid tori different from $S^1 \times S^2$. We say a lens space is nontrivial if it is not S^3.

LEMMA 3. *Let F be an orientable Seifert fiber space with $r \geqq 1$ exceptional fibers. Suppose the orbit surface is a disk. Let A be an annulus on ∂F consisting of fibers. If M is the manifold obtained from F by adding a 2-handle along A, then M is the connected sum of r nontrivial lens spaces minus an open 3-ball.*

PROOF. If $r=1$ then F is a solid torus and the theorem follows from the classification of Heegaard splittings of genus 1. Suppose the theorem is true for $r=n-1$ and suppose F has n exceptional fibers. Decompose the orbit surface f into two disks d, d' such that $\operatorname{int}(d)$ and $\operatorname{int}(d')$ contain one and $n-1$ exceptional points respectively, and $d \cap d'$ is an arc c with $c \cap \partial f = \partial c$

consisting of two points h_1, h_2 such that $h_1 \in \operatorname{int}(P(A))$, $h_2 \notin P(A)$, h_1, h_2 separate ∂f into two arcs l, l' such that $\partial d = l \cup c$, $\partial d' = l' \cup c$. Let $a = P(A) \cap d$, $a' = P(A) \cap d'$. Let $V = P^{-1}d$, $F' = P^{-1}d'$. The 2-handle B in M is decomposed into two 3-balls B_1, B_2 where the annuli $P^{-1}a$ on B_1 and $P^{-1}a'$ on B_2 constitute the annulus A on B and $B_1 \cap B_2 = D$, a disk with $\partial D = P^{-1}(h_1)$. Let $M_1 = V \cup B_1$, $M_2 = F' \cup B_2$. Clearly the hypothesis of the theorem applies to M_1, M_2; therefore M_1 is a lens space minus an open 3-ball and M_2 is the connected sum of $n-1$ lens spaces minus an open 3-ball. But $M = M_1 \cup M_2$, $M_1 \cap M_2 = \partial M_1 \cap \partial M_2 = (P^{-1}c) \cup D$, a disk. It follows that M is the disk sum of M_1 and M_2 and hence that M is $M_1 \# M_2$ minus an open 3-ball.

THEOREM 3. *Let F be an orientable Seifert fiber space with $r \geqq 1$ exceptional fibers and a disk as orbit surface. If M is a sum of F and a solid torus V, then M is a Seifert fiber space (with r or $r+1$ exceptional fibers and a 2-sphere as orbit surface) or the connected sum of r (nontrivial) lens spaces.*

PROOF. If the fibering of F cannot be extended to a fibering of M then a fiber h in ∂M is a meridian of V and therefore the theorem follows from Lemma 3.

COROLLARY 2. *If F is the complement (of a regular neighborhood in S^3) of a torus knot and if M is a sum of F and a solid torus V, then M is a Seifert fiber space (with at most three exceptional fibers and a 2-sphere as orbit surface) or the connected sum of two (nontrivial) lens spaces.*

Thus in particular any such manifold is in the "Poincaré category" (i.e. every embedded homotopy 3-cell is a 3-cell).

4. **Sums of solid tori.** We now generalize Theorem 2 as follows:

THEOREM 4. *Let M be a closed 3-manifold which is a union of three solid tori (such that the intersection of any two of them is an annulus). Then M is either a Seifert fiber space (with orbit surface a 2-sphere and at most three exceptional fibers), or the connected sum of two (nontrivial) lens spaces, or the connected sum of a lens space and $S^1 \times S^2$ or $(S^1 \times S^2) \# (S^1 \times S^2)$.*

PROOF. Refer to the proof of Theorem 2, $M = V_1 \cup V_2 \cup V_3$. If $N = V_1 \cup V_2$ is a Seifert fiber space the result follows from Theorem 3. Similarly if $V_2 \cup V_3$ or $V_1 \cup V_3$ are Seifert fiber spaces.

If $a \sim m_1$ on ∂V_1 and $a \sim m_2$ on ∂V_2 then it is not hard to see that $N = V_1 \cup V_2$ is the connected sum of a solid torus and $S^1 \times S^2$. Therefore M is the connected sum of a lens space and $S^1 \times S^2$ or $(S^1 \times S^2) \# (S^1 \times S^2)$.

5. **Unions of two solid tori.** Consider the union of two solid tori V_1, V_2 along an annulus A (which is not meridional on V_1 and V_2). Fibering V_1 and V_2 such that ∂A is a fiber, we obtain a Seifert fiber space N with invariants α_1, β_1; α_2, β_2 ($\alpha_i \geqq 2$, $0<\beta_i<\alpha_i$) and fundamental group $G=\{t_1, t_2 : t_1^{\alpha_1}=t_2^{\alpha_2}\}$. If N' is another such space with invariants α_1, β_1'; α_2, β_2' then it follows from [6] that N is not homeomorphic to N' if $\beta_i' \neq \beta_i$. Therefore (for $\alpha_1>3$) this construction gives us 3-manifolds with boundary a torus which are not homeomorphic but which have the same fundamental group G, the group of a torus knot. Such manifolds have also been obtained by H. Zieschang [8] by considering certain Heegaard splittings of genus 2.

I thank the referee for suggesting the following theorem.

THEOREM 5. *Let M be a union of two solid tori along an annulus. M is the complement* (*of a regular neighborhood in S^3*) *of a torus knot* (*including the unknot*) *if and only if M can be embedded in a simply connected* 3-*manifold N.*

PROOF. We may assume that M is not a solid torus. Therefore $\pi_1(\partial M)\to\pi_1(M)$ is an injection. Let $V=\mathrm{Cl}(N-M)$. If $\pi_1(\partial V)\to\pi_1(V)$ is an injection then $N=V\cup M$ would not be simply connected. But if $\pi_1(\partial V)\to \pi_1(V)$ is not an injection then V is a fake solid torus, i.e. V is obtained by identifying a pair of disjoint disks in the boundary of a homotopy 3-ball. For by Dehn's lemma and the Loop theorem there is a disk D in V such that $\partial D=\partial V\cap D$ is not contractible on ∂V. Cutting V along D we obtain a submanifold of N with boundary a 2-sphere, which is a homotopy 3-ball. Let $m=\partial D$. This is a simple closed curve on ∂M and $\pi_1(M)$ becomes trivial by adding a 2-handle along an annular neighborhood of m on ∂M. Filling in the resulting 2-sphere with a 3-ball we obtain a simply connected 3-manifold which is a sum of M and a solid torus. By Theorem 2, M is the complement of a regular neighborhood of a knot in S^3. Since $\pi_1(M)$ has a nontrivial center, it is a torus knot [2].

ADDED IN PROOF. A detailed version of Corollary 2 of this paper has been obtained by L. Moser [9].

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