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PREIMAGES OF POINTS UNDER THE NATURAL MAP FROM $\beta(N \times N)$ TO $\beta N \times \beta N$

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ABSTRACT. This paper deals with the size of the preimages of points of $\beta N \times \beta N$ under the continuous extension, τ, of the identity map on $N \times N$. It is concerned with those points (p, q) of $\beta N \times \beta N$ for which $\tau^{-1}(p, q)$ is infinite and extends the work of Blass [1] who thoroughly considered those points with finite preimages.

1. **Introduction.** In the construction of βN used here the points of $\beta N \backslash N$ are free ultrafilters on N. The reader is referred to the Gillman and Jerison textbook [2] for this construction and any unfamiliar terminology.

It is easily seen that if either p or q is in N then $|\tau^{-1}(p, q)| = 1$. (See Lemma 2.1 below.) In addition Blass [1] has shown the following, where p and q are in $\beta N \backslash N$. For any p, $|\tau^{-1}(p, p)| \geqq 3$ and equality holds if and only if p is a Ramsey ultrafilter. (An ultrafilter p on N is Ramsey provided whenever N is the union of pairwise disjoint subsets A_n either some A_n is in p or there is some B in p such that $|B \cap A_n| = 1$ for each n. An ultrafilter p is a P-point of $\beta N \backslash N$ provided for each countable subset $\{Z_n\}_{n=1}^{\infty}$ of p there is a member Z of p such that $Z \backslash Z_n$ is finite for each n, so in particular each Ramsey ultrafilter is a P-point of $\beta N \backslash N$.) If p and q are Ramsey and not isomorphic (i.e. there is no permutation of N whose extension to βN takes p to q) then $|\tau^{-1}(p, q)| = 2$. He also shows, assuming the continuum hypothesis, that for every integer n bigger than 1 there exist p and q with $|\tau^{-1}(p, q)| = n$. Both [3] and [1] contain the information that if $|\tau^{-1}(p, q)| = 2$ then p and q are P-points of $\beta N \backslash N$.

It is shown here that there exists a P-point p of $\beta N \backslash N$ such that $|\tau^{-1}(p, p)| = 2^c$ and that there exist distinct P-points p and q such that $|\tau^{-1}(p, q)| = 2^c$. Both results assume the continuum hypothesis. (In fact the existence of P-points of $\beta N \backslash N$ has not been shown without the aid of the continuum hypothesis.) It is also shown that there exists a point p of $\beta N \backslash N$ such that $|\tau^{-1}(p, q)| = 2^c$ for every q in $\beta N \backslash N$.

2. **Preliminary lemmas.** Lemmas 2.1 and 2.2 are well known. Their proofs are included for completeness.

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2.1. Lemma. *Let p and q be elements of βN and let $n \in N$. Then $|\tau^{-1}(p, q)| \geqq n$ if and only if there exist n pairwise disjoint subsets of $N \times N$ such that (p, q) is in the closure in $\beta N \times \beta N$ of each.*

Proof. *Necessity.* Let $\{r_i\}_{i=1}^n \subseteq \tau^{-1}(p, q)$ such that $r_i \neq r_j$ when $i \neq j$. Then there exists $\{A_i\}_{i=1}^n$ such that $A_i \in r_i$ for each i and $A_i \cap A_j = \varnothing$ when $i \neq j$. Suppose for some i that $(p, q) \notin \mathrm{cl}_{\beta N \times \beta N} A_i$. Then there is a neighborhood U of (p, q) in $\beta N \times \beta N$ such that $U \cap A_i = \varnothing$. But then $\tau^{-1}(U) \cap A_i = \varnothing$ while $\tau^{-1}(U)$ is a neighborhood of r_i in $\beta(N \times N)$.

Sufficiency. Let $\{A_i\}_{i=1}^n$ be a set of pairwise disjoint subsets of $N \times N$ such that $(p, q) \in \bigcap_{i=1}^n \mathrm{cl}_{\beta N \times \beta N} A_i$. Now $\tau(\mathrm{cl}_{\beta(N \times N)} A_i) \supseteq \mathrm{cl}_{\beta N \times \beta N} A_i$ for each i. Thus, if $i \leqq n$, one has some r_i in $\tau^{-1}(p, q) \cap \mathrm{cl}_{\beta(N \times N)} A_i$. If $i \neq j$ then $A_i \cap A_j = \varnothing$ so $r_i \neq r_j$.

2.2. Lemma. *Let p and q be elements of βN. If $|\tau^{-1}(p, q)| \geqq \aleph_0$ then $|\tau^{-1}(p, q)| = 2^c$.*

Proof. Since $|\beta(N \times N)| = 2^c$ we only need show that $|\tau^{-1}(p, q)| \geqq 2^c$. But $\tau^{-1}(p, q)$ is an infinite compact subset of the F-space $\beta(N \times N)$ so by 14N of [2] it contains a copy of βN.

In the proofs of the main theorems one constructs ultrafilters p and q on N such that (p, q) is in the closure of each of $\aleph_0$ pairwise disjoint subsets of $N \times N$. Lemmas 2.1 and 2.2 then guarantee that $|\tau^{-1}(p, q)| = 2^c$.

The author apologizes for the formalization in the following definition. Intuitively it says that A has property S if for each m there is an n such that each n-block of N contains an m-gap of A.

2.3. Definition. Let $A \subseteq N$ and let the variables m, n, z and x range over N. A has property S if $(\forall m)(\exists n)(\forall z)(\exists x)(z < x < x+m < z+n$ and $\{x, x+1, \cdots, x+m\} \cap A = \varnothing)$.

The easy proof of the following lemma is omitted.

2.4. Lemma. *If A and B have property* S *then $A \cup B$ has property* S.

2.5. Definition. Let $r \in N$. $Z(r) = \{(m, n) \in N \times N : rn < m \leqq (r+1)n\}$ and $B(r) = \bigcup \{Z(s) : s = 2^{r-1}(2m-1)$ for some m in $N\}$.

The set $\{B(r)\}_{r \in N}$ forms the countable collection of pairwise disjoint subsets of $N \times N$ referred to above.

2.6. Lemma. *Let $A \subseteq N$ and let B be an infinite subset of N. If there is some r in N such that $(A \times B) \cap B(r) = \varnothing$ then A has property* S.

Proof. Let $m \in N$ and let $w \in B$ such that $w > m$. Let $n = 2^{r+1} \cdot w$ and let $z \in N$. Let p be the least integer such that $z \leqq 2^{r-1}(2p+1)w + w$ and let

$x=2^{r-1}(2p+3)w+1$. Then

$$\begin{aligned} z &\leqq 2^{r-1}(2p+1)w+w<2^{r-1}(2p+3)w<x<x+m \\ &\leqq x+w-1=2^{r-1}(2p+3)w+w \\ &=2^{r-1}(2p-1)w+w+2^{r+1}w \\ &=2^{r-1}(2p-1)w+w+n<z+n. \end{aligned}$$

(The last inequality holds because of the choice of p.) Now let $k\in\{0,1,\cdots,m\}$. If $x+k$ were in A then $(x+k,w)$ would be in $(A\times B)\cap Z(2^{r-1}(2p+3))$ hence in $(A\times B)\cap B(r)$. Thus $\{x,x+1,\cdots,x+m\}\cap A=\varnothing$ as desired.

All the machinery needed to prove Theorem 3.1 has now been developed. The rest of this section is needed to obtain pairs of P-points of $\beta N\backslash N$ with large preimages.

2.7. Lemma. *Let $\{C_k\}_{k=1}^n$ be a set of subsets of $N\times N$ and let $A\subseteq N$. If for each finite subset F of A there exists $\{D_k\}_{k=1}^n$ such that $F=\bigcup_{k=1}^n D_k$ and, for each k, $(D_k\times D_k)\cap C_k=\varnothing$ then there exists $\{A_k\}_{k=1}^n$ such that $A=\bigcup_{k=1}^n A_k$ and, for each k, $(A_k\times A_k)\cap C_k=\varnothing$.*

Proof. If A is itself finite there is nothing to prove. Otherwise let Γ be the set of all n-tuples $(G_1,G_2,\cdots,G_n)$ such that (1) each G_k is a finite subset of A and (2) whenever F is a finite subset of A which contains $\bigcup_{k=1}^n G_k$ there is a set $\{D_k\}_{k=1}^n$ such that $\bigcup_{k=1}^n D_k=F$ and, for each k, $D_k\supseteq G_k$ and $(D_k\times D_k)\cap C_k=\varnothing$.

Partially order Γ by agreeing that $(G_1,G_2,\cdots,G_n)<(H_1,H_2,\cdots,H_n)$ provided $G_k\subseteq H_k$ for each k and the first element of $A\backslash\bigcup_{k=1}^n G_k$ is in $\bigcup_{k=1}^n H_k$.

We first note that Γ has no maximal element. Suppose instead that $(G_1,G_2,\cdots,G_n)$ is a maximal element of Γ and let a be the first element of $A\backslash\bigcup_{k=1}^n G_k$. For each j and k in $\{1,2,\cdots,n\}$ let $H_{j,k}=G_k$ if $j\neq k$ and let $H_{j,k}=G_k\cup\{a\}$ if $j=k$. If, for any j, $(H_{j,1},H_{j,2},\cdots,H_{j,n})\in\Gamma$ then $(G_1,G_2,\cdots,G_n)$ is not maximal in Γ. Consequently, for each j one has $(H_{j,1},H_{j,2},\cdots,H_{j,n})\notin\Gamma$. That is, for each j there is a finite subset F_j of A containing $\bigcup_{k=1}^n H_{j,k}$ such that if $F_j=\bigcup_{k=1}^n D_k$ and, for each k, $H_{j,k}\subseteq D_k$ then, for some k, $(D_k\times D_k)\cap C_k\neq\varnothing$. Let $F=\bigcup_{j=1}^n F_j$. Then $F\supseteq\bigcup_{k=1}^n G_k\cup\{a\}$ and F is a finite subset of A. Since $(G_1,G_2,\cdots,G_n)\in\Gamma$ there exists a set $\{D_k\}_{k=1}^n$ such that $F=\bigcup_{k=1}^n D_k$ and, for each k, $G_k\subseteq D_k$ and $D_k\times D_k\cap C_k=\varnothing$. But then, for some j, $a\in D_j$. For this j one has $F_j=\bigcup_{k=1}^n(D_k\cap F_j)$ and, for each k, $H_{j,k}\subseteq D_k\cap F_j$ and $(D_k\cap F_j)\times(D_k\cap F_j)\cap C_k=\varnothing$, contradicting the choice of F_j.

Note also that by the hypothesis of the lemma, $(\varnothing,\varnothing,\cdots,\varnothing)\in\Gamma$ so that $\Gamma\neq\varnothing$. Since the conclusion of Zorn's lemma fails and since $\Gamma\neq\varnothing$ there must be an infinite chain in Γ, say $\{(G_{j,1},G_{j,2},\cdots,G_{j,n})\}_{j=1}^\infty$

For each k in $\{1, 2, \cdots, n\}$ let $A_k = \bigcup_{j=1}^{\infty} G_{j,k}$. Then $A = \bigcup_{k=1}^{n} A_k$ since the jth element of A must be in $\bigcup_{k=1}^{n} G_{j,k}$. And $(A_k \times A_k) \cap C_k = \varnothing$ for each k since $(G_{j,k} \times G_{j,k}) \cap C_k = \varnothing$ for each j and k and since $G_{j,k} \subseteq G_{j+1,k}$.

It may be noted that in the above proof one can, by suitably restricting Γ, appeal to König's infinity lemma instead of Zorn's lemma to establish the infinite chain in Γ.

2.8. Definition. A subset A of N is sparse if there exist n in N, a set $\{A_i\}_{i=1}^{n}$ of subsets of N and a subset $\{r_i\}_{i=1}^{n}$ of N such that $A = \bigcup_{i=1}^{n} A_i$ and $(A_i \times A_i) \cap B(r_i) = \varnothing$.

Note that the finite union of sparse sets is sparse. Note also that N is not sparse. (If it were one would have $N = \bigcup_{i=1}^{n} A_i$ with $(A_i \times A_i) \cap B(r_i) = \varnothing$. Then by Lemma 2.6 each A_i would have property S and hence, by Lemma 2.4, N would have property S, which is impossible.)

2.9. Lemma. *If A is not sparse then for each finite sequence $\{r_i\}_{i=1}^{n}$ in N there is a finite subset F of A such that for each $\{D_i\}_{i=1}^{n}$ for which $F = \bigcup_{i=1}^{n} D_i$ one has some i such that $(D_i \times D_i) \cap B(r_i) \neq \varnothing$.*

Proof. Suppose there exists $\{r_i\}_{i=1}^{n}$ such that each finite subset F of A can be written $F = \bigcup_{i=1}^{n} D_i$ with $(D_i \times D_i) \cap B(r_i) = \varnothing$ for each i. Then by Lemma 2.7 there exists $\{A_i\}_{i=1}^{n}$ such that $A = \bigcup_{i=1}^{n} A_i$ and $(A_i \times A_i) \cap B(r_i) = \varnothing$ for each i. Hence A is sparse, a contradiction.

3. **Points with infinite preimages.** The first theorem provides without the benefit of the continuum hypothesis, an abundance of pairs (p, q) in $\beta N \times \beta N$ with infinite preimages.

3.1. Theorem. *There is a point p of $\beta N \backslash N$ such that $|\tau^{-1}(p, q)| = 2^c$ for every point q in $\beta N \backslash N$.*

Proof. By Lemma 2.4, the sets with property S constitute a proper ideal of subsets of N, so their complements constitute a proper filter. By extending this filter to an ultrafilter, we obtain a point p of $\beta N \backslash N$, no elements of which have property S.

Now let $q \in \beta N \backslash N$ and let $Z \in p$, $W \in q$ and $r \in N$. Then W is infinite and Z does not have property S so by Lemma 2.6 $(Z \times W) \cap B(r) \neq \varnothing$. Thus $(p, q) \in \mathrm{cl}_{\beta N \times \beta N} B(r)$ for each r in N and so, by Lemmas 2.1 and 2.2, $|\tau^{-1}(p, q)| = 2^c$.

It should be observed that, by taking $C(r) = \{(x, y) : (y, x) \in B(r)\}$, one also has that $|\tau^{-1}(q, p)| = 2^c$ where p is the point constructed above and q is any point in $\beta N \backslash N$.

It should also be noted that the point p constructed in Theorem 3.1 cannot be a P-point of $\beta N \backslash N$. Indeed any P-point of $\beta N \backslash N$ must have some element with property S. (To see this let $A_{n,k} = \{k + rn : r \in N\}$ for each n in N and k in $\{1, 2, \cdots, n\}$. Then any ultrafilter p on N has, for each n, some k such that $A_{n,k} \in p$. Let, for each n, $Z_n = A_{n,k}$ where $A_{n,k} \in p$. If p is in addition a P-point of $\beta N \backslash N$ there is a member Z of p such that $Z \backslash Z_n$ is finite for each n. This Z must have property S.)

One might then guess, since points with the smallest possible preimages must be P-points, that all pairs of P-points would have finite preimages. The following theorem shows that this is not the case, provided the continuum hypothesis is assumed.

3.2. THEOREM. *Assume the continuum hypothesis. There exists a P-point p of $\beta N \backslash N$ such that* $|\tau^{-1}(p, p)| = 2^{c}$.

PROOF. Index the subsets of N by the ordinals less than ω_1, writing $\mathscr{P}(N) = \{A_\alpha\}_{\alpha<\omega_1}$. If A_0 is sparse let $Z_0 = N \backslash A_0$. Otherwise let $Z_0 = A_0$. Let $V_0 = Z_0$ and assume that for each $\sigma < \alpha$ we have chosen Z_σ and V_σ such that: (1) $Z_\sigma = A_\sigma$ or $Z_\sigma = N \backslash A_\sigma$, (2) if $\gamma \leqq \sigma$ then $|V_\sigma \backslash Z_\gamma| < \aleph_0$ and $|V_\sigma \backslash V_\gamma| < \aleph_0$; and (3) if Γ is a finite subset of $\{Z_\gamma : \gamma \leqq \sigma\} \cup \{V_\gamma : \gamma \leqq \sigma\}$ then $\bigcap \Gamma$ is not sparse. These inductive hypotheses clearly hold when $\sigma = 0$.

If there is some finite subset Γ of $\{Z_\sigma : \sigma < \alpha\} \cup \{V_\sigma : \sigma < \alpha\}$ such that $\bigcap \Gamma \cap A_\alpha$ is sparse let $Z_\alpha = N \backslash A_\alpha$. Otherwise let $Z_\alpha = A_\alpha$. Let $\{W_n\}_{n=1}^{\infty}$ be the set of finite sequences in N where $W_n = \{r_{n,i}\}_{i=1}^{m(n)}$ and let $\{U_n\}_{n=1}^{\infty} = \{Z_\sigma : \sigma \leqq \alpha\} \cup \{V_\sigma : \sigma < \alpha\}$. Let $S_n = \bigcap_{k=1}^{n} U_k$.

Note that if Γ is a finite subset of $\{Z_\sigma : \sigma \leqq \alpha\} \cup \{V_\sigma : \sigma < \alpha\}$ then $\bigcap \Gamma$ is not sparse. To see this suppose instead there is a finite subset Γ such that $\bigcap \Gamma$ is sparse. Then necessarily $\Gamma = \Pi \cup \{Z_\alpha\}$ where $\Pi \subseteq \{Z_\sigma : \sigma < \alpha\} \cup \{V_\sigma : \sigma < \alpha\}$ by inductive hypothesis (3). But then, by the choice of Z_α, one has a finite subfamily Δ of $\{Z_\sigma : \sigma < \alpha\} \cup \{V_\sigma : \sigma < \alpha\}$ such that $\bigcap \Delta \cap A_\alpha$ is sparse and one has $Z_\alpha = N \backslash A_\alpha$. But then, letting $\Gamma' = \Delta \cup \Pi$ one concludes that $\bigcap \Gamma'$ is sparse, contradicting hypothesis (3).

Consequently each S_n is not sparse. By Lemma 2.9 for each pair of natural numbers (n, t) there exists a finite subset $F_{n,t}$ of S_t such that for each set $\{D_i\}_{i=1}^{m(n)}$ for which $F_{n,t} = \bigcup_{i=1}^{m(n)} D_i$ there is some i such that $(D_i \times D_i) \cap B(r_{n,i}) \neq \varnothing$. Let $V_\alpha = \bigcup_{t=1}^{\infty} \bigcup_{n=1}^{t} F_{n,t}$.

Hypothesis (1) is trivially satisfied. Let $\sigma \leqq \alpha$. Then $Z_\sigma \supseteq S_k$ for some k and we have that $V_\alpha \backslash Z_\sigma \subseteq \bigcup_{t=1}^{k} \bigcup_{n=1}^{t} F_{n,t}$. (If $t > k$ then $F_{n,t} \subseteq S_k \subseteq Z_\sigma$.) Therefore $|V_\alpha \backslash Z_\sigma| < \aleph_0$. Identically one sees that $|V_\alpha \backslash V_\sigma| < \aleph_0$ when $\sigma < \alpha$ so that the hypothesis (2) is satisfied.

It has already been shown that if Γ is a finite subset of $\{Z_\sigma : \sigma \leqq \alpha\} \cup \{V_\sigma : \sigma < \alpha\}$ then $\bigcap \Gamma$ is not sparse. So to complete the induction it is only required to show that for such Γ, $\bigcap \Gamma \cap V_\alpha$ is not sparse. There is some k such that $\bigcap \Gamma \supseteq S_k$ so it indeed suffices to show that $S_k \cap V_\alpha$ is not sparse.

Suppose instead that there are some $\{D_i\}_{i=1}^m$ and $\{r_i\}_{i=1}^m$ such that $S_k \cap V_\alpha = \bigcup_{i=1}^m D_i$ and $(D_i \times D_i) \cap B(r_i) = \varnothing$ for each i. Then $\{r_i\}_{i=1}^m = W_n$ for some n. Let $t = \max\{n, k\}$ and let $D_i' = D_i \cap S_t$ for each i. Then $S_t \cap V_\alpha = \bigcup_{i=1}^m D_1'$ and $(D_1' \times D_1') \cap B(r_i) = \varnothing$ for each i. But $F_{n,t} \subseteq S_t \cap V_\alpha$ so that $(D_i' \times D_i') \cap B(r_i) \neq \varnothing$ for some i, a contradiction.

Thus each of the inductive hypotheses hold and we may choose Z_α and V_α for each $\alpha < \omega_1$. Let $p = \{Z_\alpha : \alpha < \omega_1\} \cup \{V_\alpha : \alpha < \omega_1\}$. By inductive hypotheses (1) and (3), p is an ultrafilter on N, and by hypothesis, (2) p is a P-point of $\beta N \backslash N$. By hypothesis (3), $(p, p) \in \mathrm{cl}\, B(r)$ for each r in N so by Lemmas 2.1 and 2.2 $|\tau^{-1}(p, p)| = 2^c$.

The author is grateful to A. Blass for pointing out that 3.3 below is indeed a corollary to Theorem 3.2. The author's original proof involved a lemma approximately four times as complicated as Lemma 2.7.

3.3. Corollary. *There exist distinct P-points p and q of $\beta N \backslash N$ such that $|\tau^{-1}(p, q)| = 2^c$.*

Proof. Let f be a permutation of N which takes the odd numbers onto the even numbers. Let f^β be the continuous extension of f from βN to βN and let $q = f^\beta(p)$ where $p \in \beta N \backslash N$ such that $|\tau^{-1}(p, p)| = 2^c$. Then $q \neq p$ since f^β takes no point of βN to itself. For each r in N let $C(r) = \{(x, f(y)) : (x, y) \in B(r)\}$. Since $(p, p) \in \mathrm{cl}_{\beta N \times \beta N} B(r)$ for each r one has $(p, q) \in \mathrm{cl}_{\beta N \times \beta N} C(r)$ for each r. Thus by Lemmas 2.1 and 2.2 one has $|\tau^{-1}(p, q)| = 2^c$.

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