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AMALGAMATED PRODUCTS OF PROFINITE GROUPS: COUNTEREXAMPLES

LUIS RIBES

ABSTRACT. If A_1 and A_2 are profinite groups with a common closed subgroup H, the profinite amalgamated product of A_1 and A_2 over H is said to exist if A_1 and A_2 are canonically embedded in the push-out of A_1 and A_2 over H in the category of profinite groups. It is proved that such products need not exist.

Let B_1 and B_2 be profinite groups (cf. [**9**] or [**6**]) and K a common closed subgroup of B_1 and B_2. The push-out (cf. [**5**]) B of B_1 and B_2 over K in the category of profinite groups is said to be the profinite amalgamated product of B_1 and B_2 over K if the canonical (continuous) homomorphisms of B_i to B ($i=1, 2$) are 1-1, i.e., topological embeddings. One may consult [**7**] and [**8**] for a more general setting. The purpose of this note is to prove the following result.

THEOREM. *There are topologically finitely generated pro-p-groups B_1 and B_2 with common closed subgroup K, whose profinite amalgamated product does not exist.*

This answers a question posed in [**7**].

A word on notation: If A_1 and A_2 are groups with a common subgroup H we denote by $A_1 *_H A_2$ the amalgamated product of A_1 and A_2 with H amalgamated as abstract groups (cf. e.g., [**3**, p. 312]).

1. **Preliminaries.** If A is a finitely generated nilpotent group, and p a sufficiently large prime number, then $A^{p^n}=\{a^{p^n} \mid a \in A\}$ is a subgroup of A for every $n=1, 2, \cdots$ (cf. [**4**, p. 284]).

Consider the pro-p-group:

$$ {}_p\hat{A} = \operatorname{proj}\lim_n A/A^{p^n}. $$

By a theorem of K. W. Gruenberg [**2**, Theorem 2.1 (i)], $\bigcap_{n=1}^{\infty} A^{p^n}=1$, and

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so the canonical mapping of A to ${}_p\hat{A}$ is a monomorphism. We shall identify each $a \in A$ with its canonical image $(aA^p, aA^{p^2}, \cdots) \in {}_p\hat{A}$.

LEMMA 1. *Let A be a finitely generated torsion-free nilpotent group, and let H be a subgroup of A. For a prime number p let $\varphi_p: {}_p\hat{H} \to {}_p\hat{A}$ denote the canonical homomorphism. Then for all but a finite number of primes p, φ_p is a monomorphism and $A \cap \varphi_p({}_p\hat{H}) = \varphi_p(H)$.*

PROOF. If $\alpha = (h_1H^p, h_2H^{p^2}, \cdots) \in {}_p\hat{H}$, $h \in H$, $n = 1, 2, \cdots$, $h_m^{-1}h_n \in H^{p^n}$ if $n < m$, then $\varphi_p(\alpha) = (h_1A^p, h_2A^{p^2}, \cdots)$. By a result of Baumslag [**1**, p. 202] for almost all primes p, $H^{p^n} = A^{p^n} \cap H$, $n = 1, 2, \cdots$. Hence, for almost all primes p,

$$\begin{aligned} \varphi_p(\alpha) = 1 &\Rightarrow h_n \in A^{p^n}, \qquad n = 1, 2, \cdots, \\ &\Rightarrow h_n \in H^{p^n}, \qquad n = 1, 2, \cdots, \\ &\Rightarrow \alpha = 1. \end{aligned}$$

I.e., for almost all primes p, φ_p is a monomorphism. For one of these numbers p, let $a = (aA^p, aA^{p^2}, \cdots) \in {}_p\hat{A}$, $a \in A$, and

$$\varphi_p(\alpha) = (h_1A^p, h_2A^{p^2}, \cdots) \in \varphi_p({}_p\hat{H}),$$

$h_n \in H$, $n = 1, 2, \cdots$, $h_m^{-1}h_n \in H^{p^n}$ if $n < m$, and assume $a = \varphi_p(\alpha)$. Then $h_n^{-1}a \in A^{p^n}$, $n = 1, 2, \cdots$; hence $a \in \bigcap_{n=1}^{\infty} HA^{p^n}$. By Proposition 3 in [**1**], for almost all primes p, $\bigcap_{n=1}^{\infty} HA^{p^n} = H$, i.e., $a \in H$. Thus, for almost all prime p, $A \cap \varphi_p({}_p\hat{H}) \subset \varphi_p(H)$. Since the opposite inclusion obviously holds, our result follows.

LEMMA 2. *Let $A_1 \subset B_1$ and $A_2 \subset B_2$ be abstract groups, let H be a common subgroup of A_1 and A_2, and K a common subgroup of B_1 and B_2 with $H \hookrightarrow K$. Assume $A_1 \cap K = A_2 \cap K = H$. Then the canonical map φ: $A_1 *_H A_2 \to B_1 *_K B_2$ induced by $A_1 \hookrightarrow B_1$ and $A_2 \hookrightarrow B_2$ is a monomorphism.*

PROOF. Let $\{\alpha_i \mid i \in I\}$ and $\{\beta_j \mid j \in J\}$ be right transversals of H in A_1 and H in A_2 respectively. Then $\{\alpha_i \mid i \in I\}$ and $\{\beta_j \mid j \in J\}$ can be extended to right transversals of K in B_1 and K in B_2 respectively. Every element $x \in A_1 *_H A_2$ can be written uniquely in the form $x = hr_1r_2 \cdots r_n$, where $h \in H$ and $r_i \in \{\alpha_i, \beta_j \mid i \in I, j \in J\}$ [3, p. 314]. It is then clear that if $x \neq 1$ then $\varphi x = hr_1r_2 \cdots r_n \neq 1$ as an element of $B_1 *_K B_2$.

2. **Proof of the theorem.** Let A_1 and A_2 be finitely generated torsion-free nilpotent groups with a common subgroup H such that $A_1 *_H A_2$ is not residually finite (cf. [**1**, Theorem 1]). By Lemma 1, we can choose a prime number p such that if $B_1 = {}_p\hat{A}_1$, $B_2 = {}_p\hat{A}_2$, $K = {}_p\hat{H}$ then $A_1 \subset B_1$, $A_2 \subset B_2$, $H \subset K$, and $A_1 \cap K = A_2 \cap K = H$. Hence, by Lemma 2, $A_1 *_H A_2 \subset B_1 *_K B_2$, and so $B_1 *_K B_2$ cannot be residually finite. Now, let B be the

push-out of the pro-p-groups B_1 and B_2 over the common closed subgroup K, in the category of profinite groups. Assume B_1 and B_2 are canonically embedded in B (i.e., assume the profinite amalgamated product of B_1 and B_2 over K exists). Let $\{U_\mu | \mu \in M\}$ be the set of open normal subgroups of B, and put $U_{i\mu} = U_\mu \cap B_i$, $i=1, 2$. Then since K is compact, we have $\bigcap_{\mu \in M} KU_{i\mu} = 1$, $i=1, 2$. Using this fact one easily sees (cf. [**1**, Proposition 2]) that if $\mathcal{N}$ is the set of normal subgroups N of finite index in $B_1 *_K B_2$ such that $N \cap B_1$ and $N \cap B_2$ are open in B_1 and B_2 respectively, then $P = \bigcap_{N \in \mathcal{N}} N = 1$. But $P = \ker \bar{\varphi}$, where $\bar{\varphi}$ is the canonical homomorphism of $B_1 *_K B_2$ into B (cf. [7, p. 358]). Since B is a profinite group it is residually finite, and hence so is $B_1 *_K B_2$. This contradiction shows that B_1 and B_2 cannot be canonically embedded in B.

Remark. The above proof also shows that the pro-$\mathcal{C}$ amalgamated product (cf. [7]) of B_1 and B_2 over K does not exist either, where $\mathcal{C}$ is any class of finite groups containing the p-groups (for the fixed p of the theorem) closed under the formation of subgroups, direct products and homomorphic images.

Note. When this paper was already submitted for publication, J.-P. Serre sent me another example of two profinite groups with a common closed subgroup whose amalgamated product does not exist. Since his example is very different from mine I will mention its main features. I wish to thank J.-P. Serre for allowing me to include his example in this paper. Let A be a nontrivial finite group, and $T = A^{\mathbf{Z}}$, the product of an infinite number of copies of A. Define two automorphisms σ_1 and σ_2 of T by $\sigma_1((x_n)) = (x_{-n})$, $\sigma_2((x_n)) = x_{1-n}$. These are automorphisms of T of order 2, and hence they define actions of $Z/2Z$ on T, which in fact are continuous. Hence the semidirect products $G_1 = T \times_{\sigma_1} (Z/2Z)$ and $G_2 = T \times_{\sigma_2} (Z/2Z)$ are profinite groups. One shows then that T does not contain open subgroups which are normal in both G_1 and G_2. Therefore by Theorem 1.2 in [7], the amalgamated product of G_1 and G_2 over T does not exist.

It should be pointed out that this example is different from the one described in the theorem. The groups G_1, G_2, T are topologically infinitely generated, while B_1, B_2 and K are topologically finitely generated.

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DEPARTMENT OF MATHEMATICS, CARLETON UNIVERSITY, OTTAWA, ONTARIO, CANADA