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HALL-HIGMAN TYPE THEOREMS. IV

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ABSTRACT. Hall and Higman's Theorem B is proved by constructing the representation in the group algebra. This proof is independent of the field characteristic, except in one case.

Let R be an extra special r group. Suppose $C \leqq \mathrm{Aut}(R)$ is cyclic, irreducible faithful on $R/Z(R)$, and trivial on $Z(R)$. The group CR and its representation theory have been widely studied and are of some importance. Let k be a field of characteristic $q \neq r$ containing a splitting field for R. Then any faithful irreducible $k[R]$-module V is absolutely irreducible and extends to CR.

Hall and Higman [3] studied $V|_C$ when C was a q group. Their result is proved using inequalities on the dimension of $\mathrm{Hom}_{k[C]}(V, V)$. This proof also works if we only know $q \mid |C|$. The central ideal of this proof also works when $(q, |C|)=1$ [5]. However, the count is quite different. There are character proofs of the result when $(q, |C|)=1$ [1]. Thompson gave a very pretty proof of Hall and Higman's original result using vertices and sources [4].

These proofs suffer from one or more of the following difficulties:

(1) They are all tied to the characteristic q.

(2) When $q \mid |C|$ they depend upon knowing all indecomposable $k[C]$-modules.

(3) They are indirect in that they do not construct a representation of CR.

The theorem in all cases is that $V|_C$ is a direct sum of copies of the regular C-module and one other module U isomorphic to $k[C]/k[C]F$ where $F = \sum_{x \in C} x$. The number of regular modules and the appearance of U are completely independent of the field characteristic. This makes (1) a strong objection. Objection (2) becomes important if we drop the condition that C be cyclic. Finally, a "useful" construction of a given module is often better than no construction at all.

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The object of this paper is to remedy these three objections for *odd primes r*. The representation of CR on V is explicitly constructed independent of field characteristic. Then its restriction to C is studied. If $|R|=r^{2e+1}$ then it is shown that $V|_C$ contains $(r^e+1)/|C|-1$ copies of the regular C-module and one copy of $\boldsymbol{k}[C]/\boldsymbol{k}[C]F$.

I. A remark on central simple algebras.

(1.1) *Suppose $\boldsymbol{k}$ is a field and A is a central simple algebra over $\boldsymbol{k}$ of dimension t^2. Assume B is a commutative quasi-Frobenius subalgebra with* 1. *Let V be an irreducible A-module. Then $V|_B \simeq {}_BB+U$ where ${}_BB$ is B^+ considered as a left B-module and U is some complementary B-module.*

The proof is easy. Since B is quasi-Frobenius, ${}_BB$ is injective. Now $B^+\subseteq A^+$ so that ${}_AA|_B \simeq {}_BB\oplus W$ for some B-module W. The subalgebra B is commutative. So in a complete decomposition of ${}_BB$ into indecomposables, each indecomposable summand appears with multiplicity one. For some $s|t$, ${}_AA\simeq s\times V$ where V is the irreducible A-module uniquely determined up to isomorphism. Using the Krull-Schmidt theorem to compare complete decompositions of the isomorphic modules ${}_BB\oplus W$ and $s\times V|_B$ we discover that ${}_BB$ is isomorphic to a summand of $V|_B$.

(1.2) *If in* (1.1) *the dimension of B is t then $V|_B \simeq {}_BB$.*

(1.3) *Suppose $\boldsymbol{k}$ is a field and $g(x)\in \boldsymbol{k}[x]$ has positive degree. Then $\boldsymbol{k}[x]/(g(x))$ is a quasi-Frobenius algebra.*

This is well known. See [2, Section 58, Problem 2(c)].

II. A nonsingular matrix.

(2.1) *Let r be an odd prime and $\boldsymbol{K}=\mathrm{GF}(r)$. Let V be an $e\geqq 1$ dimensional vector space over $\boldsymbol{K}$ and $f: V\times V\to \boldsymbol{K}$ a nonsingular symmetric form.*

Since $\boldsymbol{Z}/r\boldsymbol{Z}\simeq \mathrm{GF}(r)$ for the integers $\boldsymbol{Z}$, we may imagine that each $a\in \boldsymbol{K}$ is a least residue in $\boldsymbol{Z}$.

(2.2) *If $\boldsymbol{k}$ is a field of characteristic $q\neq r$ containing a primitive rth root of unity ζ then the matrix $M=[\zeta^{\alpha\beta}]_{(\alpha,\beta)\in K\times K}$ is nonsingular.*

Let $N=[\zeta^{-\alpha\beta}]$. Then the (β,γ) entry of MN is

$$\sum_{\alpha}\zeta^{\beta\alpha-\alpha\gamma}=\sum_{\alpha}\zeta^{(\beta-\gamma)\alpha}=0$$

unless $\beta=\gamma$ in which case it is r. So $MN=rI$ where I is the identity. Since $r\neq 0$ in $\boldsymbol{k}$, M has inverse $r^{-1}N$.

(2.3) *If $\boldsymbol{k}$ is a field of characteristic $q\neq r$ containing a primitive rth root of unity ζ then the matrix $M=[\zeta^{f(u,v)}]_{(u,v)\in V\times V}$ is nonsingular.*

We proceed by induction on dim V. If dim $V=1$ then $f(\alpha u, \beta u)=\alpha\beta\mu$ for some $\mu\neq 0$ in $\boldsymbol{K}$ so M is the matrix of (2.2) with $\zeta\mu$ in place of ζ and (2.3) holds in this case.

Suppose dim $V=e$ and (2.3) holds for all spaces of smaller dimension. Let $\{v_1, \cdots, v_e\}$ be an orthogonal basis for V. Such a basis exists since r is odd. Let $U=\langle v_2, \cdots, v_e\rangle$. Now $f(v_1, v_1)=\mu$. Set $B=[\zeta^{\alpha\beta\mu}]$. Then we may arrange M into blocks

$$\begin{aligned} M &= [\zeta^{f(\alpha v_1+u,\beta v_1+v)}] \\ &= [\zeta^{\alpha\beta\mu+f(u,v)}] = [B\zeta^{f(u,v)}] \\ &= B \otimes [\zeta^{f(u,v)}]_{(u,v)\in U\times U}. \end{aligned}$$

So our matrix is formed as a Kronecker product. Now dim $U=e-1$ so the second matrix in the product is nonsingular. The first matrix is just the matrix of (2.2) for some primitive root; hence is nonsingular. Therefore M is nonsingular.

As a corollary we obtain the following:

(2.4) *Let* $\tilde{\boldsymbol{K}}=\mathrm{GF}(r^e)$. *Let* $\Gamma\in\tilde{\boldsymbol{K}}^\times$, $\mathrm{Tr}:\tilde{\boldsymbol{K}}\to\boldsymbol{K}$ *the trace map,* $V=\tilde{\boldsymbol{K}}^+$, *and* $f(u, v)=\mathrm{Tr}(\Gamma uv)$. *Let* ζ *be a primitive* r*th root in* $\boldsymbol{k}$ *of characteristic* $q\neq r$. *If* $\phi_u\in\boldsymbol{k}$, $u\in\tilde{\boldsymbol{K}}$ *and* $\sum_u \phi_u\zeta^{(fu,v)}=0$ *for all* $v\in\tilde{\boldsymbol{K}}$, *then all* $\phi_u=0$.

The ϕ_u's give a linear dependence on the columns of M in (2.3). So this is obvious.

III. **The group.** Let r be an odd prime and $e\geqq 1$ an integer. Let $\boldsymbol{K}=\mathrm{GF}(r)$, $\tilde{\boldsymbol{K}}=\mathrm{GF}(r^e)$, and $\hat{\boldsymbol{K}}=\mathrm{GF}(r^{2e})$. Let $\mathcal{G}$ be the Galois group of $\hat{\boldsymbol{K}}/\boldsymbol{K}$ and $\phi\in\mathcal{G}$ the element of order two in $\mathcal{G}$. Let $\mu\in\hat{\boldsymbol{K}}^+$ be of order r^e+1. Set $\nu=\mu-\mu^{-1}$. For $u, v\in\hat{\boldsymbol{K}}^+$ set

$$h(u, v) = 2^{-1}\,\mathrm{Tr}(\nu[uv^\phi - u^\phi v])$$

where $\mathrm{Tr}:\tilde{\boldsymbol{K}}\to\boldsymbol{K}$ is the trace map.

Note that $\nu[uv^\phi-u^\phi v]\in\tilde{\boldsymbol{K}}$, the fixed field of ϕ. Thus h is a nonsingular alternating form on $\hat{\boldsymbol{K}}^+$. Let $R=\hat{\boldsymbol{K}}^+\times\boldsymbol{K}^+$. For (u, ζ), $(v, \xi)\in R$ set

$$(u, \zeta)(v, \xi) = (u + v, h(u, v) + \zeta + \xi).$$

This multiplication makes R into an extra special r group of exponent r and order r^{2e+1}.

Let $C=\langle\mu\rangle$. Now C acts as automorphisms of R by

$$(u, \zeta)^x = (ux, \zeta) \quad \text{for } x\in C, \quad (u, \zeta)\in R.$$

We let $G=CR$ be the semidirect product of R by C.

If $G_0=C_0R_0$ where R_0 is a normal extra special r subgroup of G_0 with $Z(R_0)=Z(G_0)$ and C_0 is a cyclic r' group irreducible on $R_0/Z(R_0)$ then G_0 is isomorphic to a subgroup of G. This fact seems to be well known. In any case it is a straightforward computation.

IV. **The group algebra of CR.** Let k be a field of characteristic $q\neq r$ which contains a primitive rth root of 1.

We now state some facts about the group algebra of R over k. Let λ be a primitive rth root of unity in k. If $z=(0, 1)\in R$ then set

$$E = r^{-1}(\lambda^{r-1} + \lambda^{r-2}z + \cdots + \lambda z^{r-2} + z^{r-1}) \tag{4.1}$$

and

$$\chi((u, \zeta)) = 0 \quad \text{if} \quad u \neq 0,$$
$$= r^e\lambda^\zeta \quad \text{if} \quad u = 0.$$

Then E is the primitive central idempotent of $k[R]$ belonging to the irreducible character χ. Also $k[R]E$ is a central simple algebra of dimension r^{2e} over k.

Let $c\in C^\#$ and set

$$\mathcal{O}_c = \{(v, -h(vv^\phi(c-1)^{-1}, 1)) \mid v\in \hat{K}^+\}$$

and

$$K_c = -r^{-e}\sum_{x\in\mathcal{O}_c} xE. \tag{4.2}$$

(4.3) *If $x\in R$ and $c\in C^\#$ then $K_cx^c=xK_c$.*

Before starting we note a few properties of the form h. If $u, v\in\hat{K}$ then

$$h(u, v) = h(v^\phi, u^\phi) = -h(v, u) = h(uv^\phi, 1) = -h(vu^\phi, 1).$$

We let $h(u)=h(u, 1)$. There should be no confusion since the two h's are related and functions of different numbers of variables. Now $h(u)$ is a nontrivial K-linear functional from $\hat{K}$ to K.

For $x=(u, \delta)$, we compute

$$\begin{aligned}
-r^ex^{-1}K_cx^c &= \sum_v (u, \delta)^{-1}(v, -h(vv^\phi[c-1]^{-1}))(u, \delta)^cE \\
&= \sum_v (-u, -\delta)(v, -h(vv^\phi[c-1]^{-1}))(uc, \delta)E \\
&= \sum_t (t, -h([t-u(c-1)][t-u(c-1)]^\phi(c-1)^{-1} \\
&\qquad + u[t-u(c-1)]^\phi - [t-u(c-1)]u^\phi c^{-1} + uu^\phi c^{-1}))E.
\end{aligned}$$

Here we have substituted $t=v+u(c-1)$. Proceeding further,

$$= \sum (t, -h(tt^\phi(c-1)^{-1}))(0, -h(uu^\phi))E.$$

But $uu^\phi \in \tilde{K}$ so $h(uu^\phi)=0$. Thus $-r^e x^{-1}K_c x^c = -r^e K_c$. This proves (4.3). We now compute $K_c K_d$ for $c, d \in C^\#$.

$$r^{2e}K_cK_d = \sum_u (u, -h(uu^\phi(c-1)^{-1})) \sum_v (v, -h(vv^\phi(d-1)^{-1}))E$$
$$= \sum_{u,v} (u+v, -h(uu^\phi(c-1)^{-1} + vv^\phi(d-1)^{-1} - uv^\phi))E.$$

Let $t=u+v$ and compute,

$$(*) \quad = \sum_{t,v} (t, -h(tt^\phi(c-1)^{-1} + tv^\phi(c^{-1}-1)^{-1} - vt^\phi(c-1)^{-1} + vv^\phi[1 + (c-1)^{-1} + (d-1)^{-1}]))E.$$

(4.4) *For $c \in C^\#$, $K_cK_{c^{-1}}=E$.*

From our computation of K_cK_d we obtain,

$$r^{2e}K_cK_{c^{-1}} = \sum_{t,v} (t, -h(tt^\phi(c-1)^{-1} + tv^\phi(c^{-1}-1)^{-1} - vt^\phi(c-1)^{-1} + vv^\phi[1 + (c-1)^{-1} + (c^{-1}-1)^{-1}]))E$$
$$= \sum_t (t, -h(tt^\phi(c-1)^{-1})) \cdot \sum_v (0, -h(tv^\phi(c^{-1}-1)^{-1} - vt^\phi(c-1)^{-1}))E.$$

Now

$$tv^\phi(c^{-1}-1)^{-1} - vt^\phi(c-1)^{-1} = [t(c^{-1}-1)^{-1}]v^\phi - v[t(c^{-1}-1)^{-1}]^\phi.$$

Set $A(t)=t(c^{-1}-1)^{-1}$. Then the second term of the above expression is

$$\sum_v (0, -h(A(t)v^\phi - A(t)^\phi v))E.$$

But $-h(A(t)v^\phi - A(t)^\phi v)=f(v)$ is a nontrivial K-linear map of $\hat{K}^+$ onto K provided $t\neq 0$. But then the sum is

$$\sum_v (0, f(v))E = r^{2e-1} \sum_{\alpha \in K} (0, \alpha)E \quad \text{if } t \neq 0,$$
$$= r^{2e}(0,0)E \quad \text{if } t = 0.$$

Now $\sum_\alpha (0, \alpha)E = \sum_\alpha \lambda^\alpha E = 0$. So our sum is zero unless $t=0$. Then we get

$$r^{2e}K_cK_{c^{-1}} = r^{2e}E.$$

We now have the obvious corollary:

(4.5) *If $c \in C^\#$ then K_c is invertible in $k[R]E$ and has inverse equal to $K_{c^{-1}}$.*

Let $K_1=E$.

(4.6) *The map $cx \to K_c xE$ is a representation of CR in $k[R]E$.*

The collection $\{(v, 0)E | v \in \hat{K}\}$ is a k-basis for $k[R]E$. Also $k[R]E$ is a central simple algebra over k. Thus $cx \to K_c xE$ is a projective representation of CR with a factor set n by (4.3) and (4.5). Further, n is trivial on R. That is, $K_c^{-1}(v, 0)K_c = (v, 0)^c$ all $v \in \hat{K}$.

But then $K_c K_d = n(c, d)K_{cd}$. We continue our earlier computation from (*) for the case $d \neq c^{-1}$.

$$r^{2e}K_c K_d = \sum_t (t, -h(tt^\phi(cd-1)^{-1})) \cdot \sum_v (0, -h(tt^\phi[(c-1)^{-1} - (cd-1)^{-1}] + tv^\phi(c^{-1}-1)^{-1} - vt^\phi(c-1)^{-1} + vv^\phi[1 + (c-1)^{-1} + (d-1)^{-1}]))E.$$

Put the last sum equal to $A'(t)$. Then

$$= \sum_t (t, -h(tt^\phi(cd-1)^{-1}))A'(t)E$$
$$= -r^e n(c, d) \sum_t (t, -h(tt^\phi(cd-1)^{-1}))E.$$

Since $A'(t)$ is a sum of elements from $Z(R)$, $A'(t)E = a(t)E$ where $a(t) \in k$. Further, $\{(t, -h(tt^\phi(cd-1)^{-1}))E | t \in \hat{K}\}$ is a k-basis for $k[R]E$. Therefore $-r^e n(c, d) = a(t)$ for all values of t. In particular,

$$-r^e n(c, d)E = a(0)E = \sum_v (0, -h(vv^\phi[1 + (c-1)^{-1} + (d-1)^{-1}]))E.$$

The map $v \to vv^\phi$ is the norm map $N: \hat{K}^\times \to \tilde{K}^\times$. The kernel has order $r^e + 1$. Therefore

$$-r^e n(c, d)E = E + \sum_{v \in \hat{K}^\times} (0, -h(vv^\phi[1 + (c-1)^{-1} + (d-1)^{-1}]))E$$
$$= E + (r^e + 1) \sum_{u \in \tilde{K}^\times} (0, -h(u[1 + (c-1)^{-1} + (d-1)^{-1}]))E$$
$$= -r^e E + (r^e + 1) \sum_{u \in \tilde{K}} (0, -h(u[1 + (c-1)^{-1} + (d-1)^{-1}]))E.$$

Now $-h(u[1 + (c-1)^{-1} + (d-1)^{-1}]) = f(u)$ is a nontrivial K-linear map of $\tilde{K}$ to K since $d \neq c^{-1}$. Thus

$$-r^e n(c, d)E = -r^e E + (r^e + 1) \sum_u (0, f(u))E$$
$$= -r^e E + r^{e-1}(r^e + 1) \sum_{\alpha \in K} (0, \alpha)E = -r^e E.$$

Thus $n(c, d)=1$ if $d\neq c^{-1}$. By (4.4) and the definition of K_1 we know that

$$n(1, c) = n(c, 1) = n(c, c^{-1}) = n(c^{-1}, c) = 1.$$

So n is the trivial factor set and we have an ordinary representation.

$$\sum_{c\in C^{\#}} K_c = -E. \tag{4.7}$$

As usual, we compute

$$\begin{aligned} -r^e \sum_{c\in C^{\#}} K_c &= \sum_{c,v} (v, -h(vv^{\phi}(c-1)^{-1}))E \\ &= \sum_{v} (v, 0) \sum_{c} (0, -h(vv^{\phi}(c-1)^{-1}))E. \end{aligned}$$

Fix $v\neq 0$. Compute

$$\begin{aligned} -h(vv^{\phi}(c-1)^{-1}) &= -\mathrm{Tr}(2^{-1}vvv^{\phi}[(c-1)^{-1} - (c^{-1}-1)^{-1}]) \\ &= -\mathrm{Tr}(2^{-1}vv^{\phi}v(c+1)/(c-1)). \end{aligned}$$

Next we show that the map $c\to v(c+1)/(c-1)$ is a one-one map of $C^{\#}$ onto $\tilde{K}$. Note that $(v(c+1)/(c-1))^{\phi}=v(c+1)/(c-1)$ so the map is into $\tilde{K}$. Since $|C^{\#}|=r^e=|\tilde{K}|$, if the map is one-one, it is onto. So suppose

$$v\frac{c+1}{c-1} = v\frac{d+1}{d-1}, \qquad c, d \in C^{\#}.$$

Then $(c+1)(d-1)=(d+1)(c-1)$ or $2(c-d)=0$. But r is odd so $c=d$. Thus the map is one-one onto $\tilde{K}$.

Returning again to our computation

$$\begin{aligned} -r^e \sum_{c\in C^{\#}} K_c &= \sum_{v} (v, 0) \sum_{u\in\tilde{K}} (0, -\mathrm{Tr}(2^{-1}vv^{\phi}u))E \\ &= r^e(0, 0)E + \sum_{v\neq 0} (v, 0) r^{e-1} \sum_{\alpha\in K} (0, \alpha)E = r^eE. \end{aligned}$$

This proves (4.7).

(4.8) *$\{K_c | c\in C^{\#}\}$ is a linearly independent set of vectors in $k[R]E$.*

Suppose there are constants $\psi_c \in k$ so that $\sum_{c\in C^{\#}} \psi_c K_c=0$. Then

$$\begin{aligned} 0 = -r^e \sum_{c\in C^{\#}} \psi_c K_c &= \sum_{c\in C^{\#}; v\in\hat{K}} \psi_c (v, -h(vv^{\phi}(c-1)^{-1}))E \\ &= \sum_{v\in\hat{K}} (v, 0) \sum_{c} \psi_c (0, -h(vv^{\phi}(c-1)^{-1}))E. \end{aligned}$$

But $\{(v, 0)E | v \in \hat{K}\}$ is a k-basis for $k[R]E$. Therefore

$$\sum_c \psi_c(0, -h(vv^{\phi}(c-1)^{-1}))E = 0$$

for all $v \in \hat{K}$. But $v \to vv^{\phi}$ is the norm map from $\hat{K}$ to $\tilde{K}$ and is onto. So

$$\sum_c \psi_c(0, -h(\alpha(c-1)^{-1}))E = 0$$

for all $\alpha \in \tilde{K}$.

Next look at

$$\begin{aligned} h(\alpha(c-1)^{-1}) &= 2^{-1}\operatorname{Tr}(\nu\alpha[(c-1)^{-1} - (c^{-1}-1)^{-1}]) \\ &= 2^{-1}\operatorname{Tr}(\alpha\nu(c+1)/(c-1)). \end{aligned}$$

In (4.7) we saw that the map $c \to \nu(c+1)/(c-1)$ was one-one from $C^{\#}$ onto $\tilde{K}$. Let $\beta = \beta(c) = \nu(c+1)/(c-1)$. Then we may take $\psi_c = \psi'_{\beta(c)} = \psi'_{\beta}$ and

$$0 = \sum_{\beta \in \tilde{K}} \psi'_{\beta}(0, f(\alpha, \beta))E$$

for all $\alpha \in \tilde{K}$ where $f(\alpha, \beta) = -2^{-1}\operatorname{Tr}(\alpha\beta)$ is a nonsingular symmetric form from $\tilde{K} \times \tilde{K}$ to K. That is,

$$0 = \sum_{\beta \in \tilde{K}} \psi'_{\beta}\lambda^{f(\alpha,\beta)} \quad \text{for all } \alpha \in \tilde{K}.$$

By (2.4) all $\psi'_{\beta} = 0$. This proves (4.8).

(4.9) *Define $\Phi(c) = K_c$ and extend linearly to $k[C]$. Then Φ is an algebra homomorphism with kernel $k[C]F$ where $F = \sum_{x \in C} x$.*

This is an easy consequence of (4.6), (4.7), and (4.8).

(4.10) THEOREM. *Let V be an irreducible $k[CR]$-module nontrivial on $Z(R)$. Then there is a $k[C]$-module W so that*

$$V|_C \simeq (k[C]/k[C]F) \otimes_k W.$$

Let E be the primitive central idempotent of $k[R]$ such that $EV \neq (0)$. For appropriate choice of λ in (4.1), E is given there. Let B be the k-subalgebra generated by the K_c's. Let U be an irreducible $k[R]$ module with $EU \neq (0)$. Then U is a $k[CR]$ module if we let $c \in C$ act as K_c. By (1.2), (1.3) and (4.9), $U|_C \simeq {}_B B$.

So by [2, (51.7)] there is a $CR/R \simeq C$-module W such that

$$V \simeq U \otimes_k W \quad \text{or} \quad V|_C \simeq (k[C]/k[C]F) \otimes W|_C.$$

We have the following easy corollary.

(4.11) COROLLARY. *Let D be a subgroup of C and V an irreducible $k[CR]$-module faithful on $Z(R)$. Then there is a $k[D]$-module W so that*

$$V|_D \simeq (\dim W)((r^e + 1)/|D| - 1)k[D] \oplus (k[D]/k[D]F_0) \otimes W$$

where $F_0 = \sum_{x \in D} x$.

We have completed the proof of the theorem. This construction need not be confined to k. With modification it gives a representation of CR in $\mathcal{O}$, the ring of p-adic integers $p \neq r$, in k when k is an algebraic number field.

Is there a characteristic free proof for $r=2$?

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