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GLOBAL DIMENSION OF TRIANGULAR ORDERS OVER A DISCRETE VALUATION RING

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ABSTRACT. We characterize triangular R-orders of finite global dimension in $n\times n$ matrix rings over the quotient field of DVR R and obtain a precise upper bound for their global dimension, viz. $n-1$. We also characterize triangular R-orders of highest global dimension.

Introduction. Throughout R is a discrete valuation ring (DVR) with the unique maximal ideal $\mathfrak{m}$, generated by t, and quotient field K. An R-*order* Λ in $M_n(K)$ is an R-subalgebra of $M_n(K)$ which is finitely generated as an R-module and spans $M_n(K)$ over K. Λ is *tiled* if Λ contains n orthogonal idempotents. After a conjugation, if necessary, we may assume that $e_{ii}\in\Lambda$, where e_{ij} are the usual matrix units in $M_n(K)$. Then Λ is of the form $\Lambda=(\mathfrak{m}^{\lambda_{ij}})$, where $\lambda_{ij}\in \boldsymbol{Z}$. If $\lambda_{ij}=0$ whenever $i\leqq j$ then Λ is called a *triangular* R-order. We set $\Omega_n=(\mathfrak{m}^{\mu_{ij}})\subseteq M_n(K)$, where $\mu_{ij}=0$ whenever $i\leqq j$ and $\mu_{ij}=i-j$ otherwise.

The main result of this paper is the following

THEOREM. *Given a triangular R-order in $M_n(K)$, the following are equivalent*: (1) gl. dim. $\Lambda<\infty$, (2) $\Omega_n\subseteq\Lambda$, (3) gl. dim. $\Lambda\leqq n-1$.

This result was conjectured by R. B. Tarsey [5]. In the same paper, Tarsey constructs a triangular R-order in $M_n(K)$ of global dimension $n-1$. Hence the bound in our theorem is best possible. We also give a characterization of triangular R-orders of highest global dimension. Using this we construct examples of successive triangular R-orders in $M_{2n+1}(K)$ whose global dimensions differ exactly by n. This disproves a conjecture in R. B. Tarsey [5].

The main results of this paper were announced in [1].

After this paper was completed the author received a preprint of [6] from R. B. Tarsey in which he has independently obtained (1)$\Leftrightarrow$(2) in the above theorem; however, his methods yield a bound $2n-4$ rather than $n-1$.

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The proof of the theorem depends on the following lemmas.

LEMMA 1. *If* Λ *is an R-order in an algebra over the quotient field of DVR R and if* $J(\Lambda)$ *is the Jacobson radical of* Λ, *then* gl. dim. $\Lambda=1+\mathrm{hd}_\Lambda J(\Lambda)$.

PROOF. Silver [**4**, Corollary 4.6].

LEMMA 2. *If* Λ *is any ring, then* rt. gl. dim. $\Lambda=1+\sup_I\{\mathrm{hd}\ I\}$ *where supremum is taken over right ideals of* Λ, *unless* Λ *is semisimple.*

PROOF. Well known [**3**].

Henceforth, we shall fix a positive integer n and unless stated otherwise, $\Lambda=(\mathfrak{m}^{\lambda_{ij}})$ will denote a triangular R-order in $M_n(K)$; P_i and J_i denote the ith row of Λ and its Jacobson radical respectively. We shall always treat P_i and J_i as canonical submodules of the first row of Λ. This makes expressions like P_i+P_j unambiguous. Observe that if $\lambda_{i,i-1}\gneqq 0$ for $2\leqq i\leqq n$, then $J(\Lambda)$ is obtained from Λ by replacing the diagonal entries R by $\mathfrak{m}$.

Let $e=\sum_{i=1}^{n-1} e_{ii}$, where e_{ii} are the usual idempotents in Λ. We shall interchangeably treat $e\Lambda e$ as top $(n-1)\times(n-1)$ corner of Λ or as a triangular order in $M_{n-1}(K)$. Let $\mathscr{F}$: mod-$\Lambda\to$mod-$e\Lambda e$ and $\mathscr{G}$: mod-$e\Lambda e\to$ mod-Λ be the functors defined by $\mathscr{F}(M)=Me$, $M\in$ mod-Λ and $\mathscr{G}(N)=N\otimes_{e\Lambda e}e\Lambda$ where $e\Lambda=\bigoplus_{i=1}^{n-1}P_i$. We shall have repeated occasions to use these functors.

LEMMA 3. (a) $\mathscr{F}$ *and* $\mathscr{G}$ *are exact additive functors.* $\mathscr{F}P_1,\cdots,\mathscr{F}P_{n-1}$ *are principal projectives of* $e\Lambda e$. $e\Lambda$ *is a progenerator in* $e\Lambda e$-mod *and* $J(e\Lambda e)$ *is canonically isomorphic with* $\bigoplus_{i=1}^{n-1}\mathscr{F}J_i$.

(b) $\mathscr{G}\mathscr{F}P_i\simeq P_i$ *for* $1\leqq i\leqq n-1$.

(c) *For every right* $e\Lambda e$-*module* N *and right* Λ-*module* M, *we have,*

$$\mathrm{hd}_\Lambda\,\mathscr{G}N\leqq \mathrm{hd}_{e\Lambda e}\,N,\qquad \mathrm{hd}_{e\Lambda e}\,\mathscr{F}\mathscr{G}\mathscr{F}M\leqq \mathrm{hd}_\Lambda\,\mathscr{G}\mathscr{F}M.$$

Further, if $\mathscr{G}\mathscr{F}M\simeq M$ *then* $\mathrm{hd}_\Lambda M=\mathrm{hd}_{e\Lambda e}\,\mathscr{F}M$.

PROOF. The first part is clear. For part (b) we observe that ${}_{e\Lambda e}e\Lambda$ is projective (hence flat) and $\mathscr{F}P_i$ is isomorphic to a right ideal of $e\Lambda e$, therefore

$$0\to\mathscr{F}P_i\otimes_{e\Lambda e}e\Lambda\to e\Lambda e\otimes_{e\Lambda e}e\Lambda$$

is exact. Hence $\mathscr{G}\mathscr{F}P_i\simeq(\mathscr{F}P_i)e\Lambda$. The last two entries in P_i are equal for $1\leqq i\leqq n-1$. So, $(\mathscr{F}P_i)e\Lambda=P_i$.

The first inequality in (c) is clear since $e\Lambda_\Lambda$ is projective. Now, suppose $\mathrm{hd}_\Lambda\,\mathscr{G}\mathscr{F}M=l<\infty$. Let

$$\cdots\longrightarrow M_i\xrightarrow{d_i}M_{i-1}\longrightarrow\cdots\longrightarrow M_1\xrightarrow{d_1}M_0\xrightarrow{d_0}\mathscr{F}M\longrightarrow 0$$

be a projective resolution of $\mathscr{F}M$ over $e\Lambda e$. Since $e\Lambda e$ is semiperfect and $\mathscr{F}P_1, \cdots, \mathscr{F}P_{n-1}$ are the only principal projectives of $e\Lambda e$, therefore each $M_i \simeq \bigoplus_{j=1}^{n-1} \mathscr{F}P_j^{k_{ij}}$, where the k_{ij} are (possibly empty) sets. Clearly,

$$\cdots \longrightarrow \mathscr{G}M_i \xrightarrow{d_i \otimes 1} \mathscr{G}M_{i-1} \xrightarrow{d_{i-1} \otimes 1} \cdots \longrightarrow$$
$$\mathscr{G}M_1 \xrightarrow{d_1 \otimes 1} \mathscr{G}M_0 \xrightarrow{d_0 \otimes 1} \mathscr{G}\mathscr{F}M \longrightarrow 0$$

is a projective resolution of $\mathscr{G}\mathscr{F}M$ over Λ. Since $\mathrm{hd}_\Lambda \mathscr{G}\mathscr{F}M = l < \infty$, therefore $\mathscr{G}M_l \simeq \mathrm{Im}(d_l \otimes 1) \oplus L$ for some right Λ-module L. Since $\mathscr{G}M_l \simeq \bigoplus_{j=1}^{n-1} P_j^{k_{lj}}$ and Λ is semiperfect, therefore by the decomposition theorem [**2**, Theorem 3] and Krull-Schmidt-Azumaya theorem, $\mathrm{Im}(d_l \otimes 1) \simeq \bigoplus_{j=1}^{n-1} P_j^{k'_{lj}}$ for some (possibly empty) sets k'_{lj}. This shows that $\mathscr{F}\,\mathrm{Im}(d_l \otimes 1)$ is a right $e\Lambda e$-projective module. Now,

$$0 \to \mathscr{F}\,\mathrm{Im}(d_l \otimes 1) \to \mathscr{F}\mathscr{G}M_{l-1} \to \cdots \to \mathscr{F}\mathscr{G}M_0 \to \mathscr{F}\mathscr{G}\mathscr{F}M \to 0$$

is a projective resolution of $\mathscr{F}\mathscr{G}\mathscr{F}M$ over $e\Lambda e$, which yields

$$\mathrm{hd}_{e\Lambda e}\, \mathscr{F}\mathscr{G}\mathscr{F}M \leqq l.$$

The last assertion follows from above two inequalities.

LEMMA 4. *If* $\lambda_{i,i-1} \gneqq 0$ *for* $2 \leqq i \leqq n$ *and* $\lambda_{n,n-1} = 1$, *then* $\mathscr{G}\mathscr{F}J_i \simeq J_i$ *for* $i \neq n-1$.

PROOF. The proof is similar to that of part (b) of Lemma 3.

LEMMA 5. *If* gl. dim. $\Lambda < \infty$, *then* $\lambda_{2,1} \leqq 1$ *and* $\lambda_{n,n-1} \leqq 1$.

PROOF. Suppose $\lambda_{2,1} \geqq 2$. We have exact sequences,

$$0 \longrightarrow tP_1 \cap P_2 \xrightarrow{\phi_1} tP_1 \oplus P_2 \xrightarrow{\theta_1} J_1 \longrightarrow 0,$$

$$0 \longrightarrow t^{\lambda_{2,1}-1}P_1 \cap P_2 \xrightarrow{\phi_2} t^{\lambda_{2,1}-1}P_1 \oplus P_2 \xrightarrow{\theta_2} t^{\lambda_{2,1}-1}P_1 + P_2 \longrightarrow 0,$$

where $\phi_i(x) = (x, x)$ and $\theta_i(x, y) = x - y$ for $i = 1, 2$. Since J_1 is not projective, $tP_1 \cap P_2 \simeq t^{\lambda_{2,1}-1}P_1 + P_2$ and $t^{\lambda_{2,1}-1}P_1 \cap P_2 \simeq J_1$, therefore $\mathrm{hd}_\Lambda J_1 = \infty$, contrary to our hypothesis. Hence $\lambda_{2,1} \leqq 1$. Similarly, looking at appropriate left Λ-modules we get $\lambda_{n,n-1} \leqq 1$.

LEMMA 6. (a) *If* $\Omega_n \subseteq \Lambda$, *then* $\Omega_{n-1} \subseteq e\Lambda e$.

(b) *If* $\lambda_{l,l-1} = 0$ *for some* l *and if* $f = \sum_{i=1; i \neq l}^{n} e_{ii}$, *then* $\Omega_n \subseteq \Lambda$ *iff* $\Omega_{n-1} \subseteq f\Lambda f \subseteq M_{n-1}(K)$.

PROOF. The first part is clear since $\Omega_{n-1} = e\Omega_n e$. Now assume $\Omega_n \subseteq \Lambda$.

Since

$$\lambda_{i,j} \leqq \lambda_{i,l} + \lambda_{l,l-1} + \lambda_{l-1,j} = \lambda_{i,l} + \lambda_{l-1,j},$$

therefore, if $i \gneqq l \gneqq j$ then $\lambda_{i,j} \leqq (i-l)+(l-1-j)=i-1-j$. It follows that $\Omega_{n-1} \subseteq f\Lambda f$. The remaining case is similarly dealt with.

PROPOSITION 1. *If* $\lambda_{i,i-1} \gneqq 0$ *for* $2 \leqq i \leqq n$ *and if* $\lambda_{n,n-1}=1$, *then* gl. dim. $\Lambda < \infty$ *if and only if* gl. dim. $e\Lambda e < \infty$. *Further, if* gl. dim. $\Lambda = \alpha < \infty$ *and if* gl. dim $e\Lambda e = \beta < \infty$ *then* $\beta \leqq \alpha \leqq \beta + 1$.

PROOF. By Lemma 4, $J_i \cong \mathscr{G}\mathscr{F}J_i$ for $i \neq n-1$. Clearly, $\mathscr{F}\mathscr{G}\mathscr{F}J_{n-1} \cong \mathscr{F}J_{n-1}$. Therefore, by Lemma 3,

$$\operatorname{hd}_\Lambda J_i = \operatorname{hd}_{e\Lambda e} \mathscr{F}J_i \quad \text{for } i \neq n-1$$

and $\operatorname{hd}_\Lambda (\mathscr{F}J_{n-1})e\Lambda = \operatorname{hd}_\Lambda \mathscr{G}\mathscr{F}J_{n-1} = \operatorname{hd}_{e\Lambda e} \mathscr{F}J_{n-1}$. Clearly, $(\mathscr{F}J_{n-1})e\Lambda + P_n = J_{n-1}$, $(\mathscr{F}J_{n-1})e\Lambda \cap P_n = J_n$. Hence,

$$(*) \qquad 0 \longrightarrow J_n \xrightarrow{\phi} (\mathscr{F}J_{n-1})e\Lambda \oplus P_n \xrightarrow{\theta} J_{n-1} \longrightarrow 0$$

is exact where $\phi(x)=(x, x)$ and $\theta(x, y)=x-y$. Since $\mathscr{F}J_n$ and $(\mathscr{F}J_{n-1})e\Lambda$ are isomorphic to right ideals of $e\Lambda e$ and Λ respectively, therefore Lemmas 1 and 2 complete the proof.

THEOREM 1. *Let* $\Lambda = (m^{\lambda_{ij}})$ *be a triangular order in* $M_n(K)$. *Then the following are equivalent*: (1) gl. dim. $\Lambda < \infty$, (2) $\Omega_n \subseteq \Lambda$, (3) gl. dim. $\Lambda \leqq n-1$.

PROOF. (1)$\Rightarrow$(2). Proceed by induction on n. For $n=2$, the result is trivial and known [5]. Assume $n>2$. If $\lambda_{i,i-1} \gneqq 0$ for $2 \leqq i \leqq n$ then, by Lemma 5, $\lambda_{n,n-1}=1$. So, Proposition 1 shows that gl. dim. $e\Lambda e < \infty$. Hence by induction hypothesis $\Omega_{n-1} \subseteq e\Lambda e$. Since $\lambda_{n,n-1}=1$, therefore $\Omega_n \subseteq \Lambda$. If $\lambda_{l,l-1}=0$ for some l, then Λ is Morita equivalent to $f\Lambda f$, where $f=\sum_{i=1;i\neq l}^{n} e_{ii}$. By induction hypothesis $\Omega_{n-1} \subseteq f\Lambda f \subseteq M_{n-1}(K)$, so $\Omega_n \subseteq \Lambda$ by Lemma 6.

(2)$\Rightarrow$(3). Again we put an induction on n. For $n=2$, the result is true and trivial [5]. Let $n>2$. If $\lambda_{i,i-1} \gneqq 0$ for $2 \leqq i \leqq n$, then Lemma 6 and the induction hypothesis show that gl. dim. $e\Lambda e \leqq n-2$. By Proposition 1, gl. dim. $\Lambda \leqq n-1$. If $\lambda_{l,l-1}=0$ for some l, then Λ is Morita equivalent to $f\Lambda f$. By Lemma 6, $\Omega_{n-1} \subseteq f\Lambda f \subseteq M_{n-1}(K)$, so, by induction hypothesis, gl. dim. $\Lambda =$ gl. dim. $f\Lambda f \leqq n-2$.

(3)$\Rightarrow$(1). Clear. This completes the proof.

PROPOSITION 1′. *If* $\lambda_{i,i-1} \gneqq 0$ *for* $2 \leqq i \leqq n$ *and if* $\lambda_{2,1}=1$ *then* gl. dim. $\Lambda < \infty$ *if and only if* gl. dim. $e'\Lambda e' < \infty$ *where* $e' = \sum_{i=2}^{n} e_{ii}$. *Further, if* gl. dim. $\Lambda = \alpha < \infty$, gl. dim. $e'\Lambda e' = \gamma < \infty$, *then* $\gamma \leqq \alpha \leqq \gamma + 1$.

PROOF. Similar to Proposition 1.

We now look at the triangular orders in $M_n(K)$ with global dimension $n-1$.

LEMMA 7. *Let* $\Lambda=(\mathscr{m}^{\lambda_{ij}})$ *be a triangular order in* $M_n(K)$, $n\geqq 3$. *If* gl. dim. $\Lambda=n-1$ *then* $\mathrm{hd}_\Lambda J_i=i-1$ *for* $1\leqq i\leqq n-1$ *and* $\mathrm{hd}_\Lambda J_n=n-3$.

PROOF. By induction on n. For $n=3$, the only triangular order of global dimension two is Ω_3, for which the assertion is trivial. Let $n\gneqq 3$. Since gl. dim. $\Lambda=n-1$, therefore by Theorem 1, $\lambda_{i,i-1}=1$ for $2\leqq i\leqq n$. Hence, by Proposition 1 and Theorem 1, gl. dim. $e\Lambda e=n-2$. As seen in Lemma 3 and Proposition 1, $J(e\Lambda e)\simeq\oplus_{i=1}^{n-1}\mathscr{F}J_i$, $\mathrm{hd}_\Lambda J_i=\mathrm{hd}_{e\Lambda e}\mathscr{F}J_i$ if $i\neq n-1$ and $\mathrm{hd}_\Lambda(\mathscr{F}J_{n-1})e\Lambda=\mathrm{hd}_{e\Lambda e}\mathscr{F}J_{n-1}$; by induction hypothesis, $\mathrm{hd}_\Lambda J_i=\mathrm{hd}_{e\Lambda e}\mathscr{F}J_i$ $=i-1$ for $1\leqq i\leqq n-2$, $\mathrm{hd}_\Lambda(\mathscr{F}J_{n-1})e\Lambda=\mathrm{hd}_{e\Lambda e}\mathscr{F}J_{n-1}=n-4$. Since $\mathscr{F}J_n$ is isomorphic to a right ideal of $e\Lambda e$, therefore $\mathrm{hd}_\Lambda J_n=\mathrm{hd}_{e\Lambda e}\mathscr{F}J_n\leqq n-3$, by Lemma 2. By hypothesis gl. dim. $\Lambda=n-1$. Hence, by Lemma 1, $\mathrm{hd}_\Lambda J_{n-1}=n-2$. So, by (*) in Proposition 1, $\mathrm{hd}_\Lambda J_n=n-3$. This completes the proof.

THEOREM 2. *Let* $\Lambda=(\mathscr{m}^{\lambda_{ij}})$ *be a triangular order in* $M_n(K)$, *where* $n\geqq 4$. *Then* gl. dim. $\Lambda=n-1$ *if and only if* $\lambda_{i,i-1}=1$, $\lambda_{i,i-2}=2=\lambda_{i,i-3}$ *for* $2\leqq i\leqq n$.

PROOF. For the "only if" part, we proceed by induction on n. Let $n=4$. Since gl. dim. $\Lambda=3$, therefore by Theorem 1, $\lambda_{i,i-1}=1$ for $2\leqq i\leqq 4$. By Propositions 1, 1′ and Theorem 1, gl. dim. $e\Lambda e=$gl. dim. $e'\Lambda e'=2$. Hence $\lambda_{3,1}=2=\lambda_{4,2}$, $\lambda_{4,1}=2$ or 3. But gl. dim. $\Omega_4=2$ [5]; so, we must have $\lambda_{4,1}=2$. Now let $n\gneqq 4$. Since gl. dim. $\Lambda=n-1$, therefore by Theorem 1 and Propositions 1, 1′, gl. dim. $e\Lambda e=$gl. dim. $e'\Lambda e'=n-2$. Now the induction hypothesis completes the proof.

For the "if" part again we put induction on n. The assertion is easily seen to be true for $n=4$. Now let $n\gneqq 4$. By induction hypothesis, we have gl. dim. $e\Lambda e=n-2$. So, Lemma 7 and its proof yield $\mathrm{hd}_\Lambda J_i=i-1$ for $1\leqq i\leqq n-2$, $\mathrm{hd}_\Lambda J_n\leqq n-3$ and $\mathrm{hd}_\Lambda(\mathscr{F}J_{n-1})e\Lambda=n-4$. Hence, by the exact sequence (*) it is enough to prove that $\mathrm{hd}_\Lambda J_n=n-3$.

Let M be the right Λ-module obtained from P_n by replacing the last two entries by $\mathscr{m}^2$. By hypothesis $\lambda_{n,n-2}=2=\lambda_{n,n-3}$. So the last four entries in M are equal, viz. $\mathscr{m}^2$. Clearly, as in Lemma 3(b), $\mathscr{G}\mathscr{F}M\simeq M$, so by Lemma 3, $\mathrm{hd}_\Lambda M=\mathrm{hd}_{e\Lambda e}\mathscr{F}M\leqq n-3$. The last inequality follows by observing that $\mathscr{F}M$ is isomorphic to a right ideal of $e\Lambda e$. Repeating this two more times, we get $\mathrm{hd}_\Lambda M\leqq n-5$. Clearly $M+tP_{n-1}=J_n$ and $M\cap tP_{n-1}\simeq(\mathscr{F}J_{n-1})e\Lambda$. By the above, $\mathrm{hd}_\Lambda(\mathscr{F}J_{n-1})e\Lambda=n-4$. Hence, $\mathrm{hd}\, J_n=n-3$. This completes the proof.

Now, we give examples of successive triangular orders in $M_{2n+1}(K)$ whose dimensions differ exactly by n. This disproves a conjecture of R. B.

Tarsey [5]. We define two families Λ_{2n+1} and Γ_{2n+1}, $n \geqq 1$, of triangular orders in $M_{2n+1}(K)$ such that Λ_{2n+1} and Γ_{2n+1} are successive, gl. dim. $\Lambda_{2n+1} = n$ and gl. dim. $\Gamma_{2n+1} = 2n$.

For $n=1$,

$$\Lambda_3 = \begin{pmatrix} R & R & R \\ m & R & R \\ m & m & R \end{pmatrix}, \qquad \Gamma_3 = \begin{pmatrix} R & R & R \\ m & R & R \\ m^2 & m & R \end{pmatrix}.$$

For $n=2$,

$$\Lambda_5 = \begin{pmatrix} R & R & R & R & R \\ m & R & R & R & R \\ m^2 & m & R & R & R \\ m^2 & m & m & R & R \\ m^3 & m^2 & m^2 & m & R \end{pmatrix}, \qquad \Gamma_5 = \begin{pmatrix} R & R & R & R & R \\ m & R & R & R & R \\ m^2 & m & R & R & R \\ m^2 & m^2 & m & R & R \\ m^3 & m^2 & m^2 & m & R \end{pmatrix}.$$

It is easy to see that gl. dim. $\Lambda_3 = 1$, gl. dim. $\Gamma_3 = 2$, gl. dim. $\Lambda_5 = 2$, gl. dim. $\Gamma_5 = 4$, Λ_3 and Γ_3 are successive and Λ_5 and Γ_5 are successive.

For $n \geqq 3$, let U_n be a triangular order in $M_n(K)$ in which all the entries on the main subdiagonal are m and all the entries below the main subdiagonal are m^2. Let $V_{n,n+1} = (m^{\theta_{ij}})$, where θ_{ij} are as specified below:

(a) $\theta_{1,n} = \theta_{1,n+1} = 1$; $\theta_{i,n} = \theta_{i,n+1} = 2$ for $2 \leqq i \leqq n$.

(b) $\theta_{1,n-1} = 2$; $\theta_{i,n-1} = 3$ for $2 \leqq i \leqq n$.

(c) $\theta_{1,j} = 3$ for $1 \leqq j \leqq n-2$.

(d) All the remaining $\theta_{i,j} = 4$.

Let

$$\Lambda_{2n+1} = \begin{pmatrix} U_{n+1} & M_{n+1,n}(R) \\ V_{n,n+1} & U_n \end{pmatrix}$$

and Γ_{2n+1} is obtained from Λ_{2n+1} by replacing $(n+2, n)$th entry m by m^2. Trivially, Λ_{2n+1} and Γ_{2n+1} are successive.

By Theorem 2, gl. dim. $\Gamma_{2n+1} = 2n$. We claim gl. dim. $\Lambda_{2n+1} = n$. Let P_i and J_i denote the ith row of Λ_{2n+1} and its Jacobson radical. Clearly, $J_1 \cong P_2$. Hence, hd $J_1 = 0$. Since

$$(\#) \qquad tP_{i-1} + P_{i+1} = J_i, \qquad tP_{i-1} \cap P_{i+1} \cong J_{i-1} \quad \text{for } 2 \leqq i \leqq n,$$

therefore, by induction it follows that hd $J_i = i-1$ for $2 \leqq i \leqq n$. Since $tP_n + P_{n+3} = J_{n+2}$, $tP_n \cap P_{n+3} \cong P_{n+2}$, therefore hd $J_{n+2} = 1$. Now observing that $(\#)$ holds for $n+3 \leqq i \leqq 2n$, we get, by induction, hd $J_i = i-n-1$ for $n+3 \leqq i \leqq 2n$.

Let $M_i = tP_1 + P_{i+1}$ for $2 \leqq i \leqq n-1$. Clearly $tP_1 \cap P_{i+1} \cong M_{i-1}$ for $3 \leqq i \leqq n-1$ and $M_2 = J_2$. Hence, by induction, hd $M_i = i-1$ for $2 \leqq i \leqq n-1$. But $tM_{n-1} + P_{n+2} = J_{n+1}$ and $tM_{n-1} \cap P_{n+2} \cong P_n$. Therefore, hd $J_{n+1} = n-2$.

Let $N_i = tP_n + P_{n+i}$ for $3 \leqq i \leqq n$. It is easy to see that $tP_n \cap P_{n+i} \cong N_{i-1}$ for $4 \leqq i \leqq n$ and $N_3 = J_{n+2}$. Therefore, by induction, hd $N_i = i - 2$ for $3 \leqq i \leqq n$. Since $J_{2n+1} \cong N_n$, therefore hd $J_{2n+1} = n - 2$. Hence, hd $J(\Lambda_{2n+1}) = n - 1$. Therefore, by Lemma 1, gl. dim. $\Lambda_{2n+1} = n$. This completes the proof of our claim.

REMARK. Using the usual arguments about localization and completion, it is easy to see that our results hold when R is a Dedekind domain rather than DVR.; cf. [**5**].

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