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OSCILLATORY SOLUTIONS FOR A GENERALIZED SUBLINEAR SECOND ORDER DIFFERENTIAL EQUATION

J. W. HEIDEL[1] AND I. T. KIGURADZE

ABSTRACT. A criterion is given for the existence of oscillatory solutions for equation (1) below which generalizes a recent result for the sublinear case of (1′). The present theorem is the analogue of a result of Izjumova for the generalized superlinear case.

We consider the question of the existence of oscillatory solutions of the equation

(1) $$u'' + f(t, u) = 0$$

where the function $f(t, u)$ is defined and continuous in the region $0 \leqq t < \infty$, $-\infty < u < \infty$, and $f(t, 0) \equiv 0$.

Equation (1) is a generalization of

(1′) $$u'' + q(t)u^{\gamma} = 0$$

which is called superlinear if $\gamma > 1$ and sublinear if $0 < \gamma < 1$. A criterion for the existence of oscillatory solutions for (1′) in the superlinear case was first given by Jasny [**6**] and Kurzweil [**8**]. A short proof of the Jasny-Kurzweil theorem was given by the second author [**7**]. The theorem was then generalized to (1) in several directions, first by Izjumova [**5**] and then by Coffman and Wong [**2**], [**3**].

The analogue of the Jasny-Kurzweil result for the sublinear case has recently been established by Hinton and the first author [**4**] and Chiou [**1**]. The purpose of the present note is to generalize this result by giving the analogue of Izjumova's theorem.

THEOREM. *Suppose that for every fixed $x > 0$, the function*

(2) $$\phi(t, x) = t^{3/2} f(t, t^{1/2} x)$$

is nonnegative, continuously differentiable, and nondecreasing in t in the

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interval $[t_0, \infty)$ *where* $t_0>0$. *If, moreover,* $\phi(t, -x)=-\phi(t, x)$ *and*

$$\liminf_{x\to 0}\,(\phi(t_0, x)/x) > \tfrac{1}{4}, \tag{3}$$

then equation (1) *has at least one nonsingular oscillatory solution.*

PROOF. Let $\Phi(t, x)=2\int_0^x \phi(t, s)\, ds$.

In view of (3), positive constants γ and δ can be found such that

$$\phi(t_0, x) > x(1+\gamma)/4 \quad \text{for } 0 < x \leqq \delta \tag{4}$$

and

$$\Phi(t_0, x) > x^2(1+\gamma)/4 \quad \text{for } 0 < x \leqq \delta. \tag{5}$$

Let $u(t)$ be a solution of equation (1) which satisfies the initial condition

$$u(t_0) = 0, \qquad 0 < t_0 u'^2(t_0) < \delta^2\gamma/4, \tag{6}$$

at t_0.

Multiplying both sides of equation (1) by $t^{3/2}(t^{-1/2}u(t))'$ and integrating from t_0 to t, we obtain

$$(\sqrt{t}u' - u/2\sqrt{t})^2 + \Phi(t, v(t)) - \tfrac{1}{4}v^2(t) = t_0 u'^2(t_0) + \int_{t_0}^{t} \frac{\partial\Phi(\tau, v(\tau))}{\partial\tau}\, d\tau, \tag{7}$$

where $v(t)=t^{-1/2}|u(t)|$.

Let $w(t)=\max\{v(s): t_0\leqq s\leqq t\}$.

Since $\partial\Phi(t, x)/\partial t$ is nondecreasing with respect to x in the interval (t_0, ∞) we have

$$\int_{t_0}^{t} \frac{\partial\Phi(\tau, v(\tau))}{\partial\tau}\, d\tau \leqq \int_{t_0}^{t} \frac{\partial\Phi(\tau, w(t))}{\partial\tau}\, d\tau = \Phi(t, w(t)) - \Phi(t_0, w(t)). \tag{8}$$

Therefore from (6), (7) and (8) it follows that

$$\Phi(t_0, w(t)) - \tfrac{1}{4}w^2(t) < \frac{\delta^2\gamma}{4} + \Phi(t, w(t)) - \Phi(t, v(t)). \tag{9}$$

Since $w(t_0)=0$ and $w(t)\geqq 0$ for $t\geqq t_0$, it is clear that

$$0 \leqq w(t) < \delta \tag{10}$$

in some right neighborhood of t_0. We will now show that (10) holds for all $t\geqq t_0$. Suppose to the contrary that there is a $t_1>t_0$ such that $w(t_1)=\delta$ and that t_1 is the smallest such value of t. Then $w(t_1)=v(t_1)$ and so from (5) and (9) it follows that

$$(\gamma/4)w^2(t_1) < \delta^2\gamma/4,$$

a contradiction.

Consequently

$$t^{-1/2}\,|u(t)| = v(t) < \delta \quad \text{for } t \geqq t_0. \tag{11}$$

Thus we have proven that $u(t)$ is extendable on the whole interval $[t_0, \infty)$ and satisfies the inequality (11). On the other hand, from (7) it is clear that $|u(t)|+|u'(t)|\neq 0$ for $t\geqq t_0$. Consequently, $u(t)$ is a nonsingular solution.

We will prove that $u(t)$ is oscillatory. Suppose to the contrary that for some $t^*>t_0$, $u(t)\neq 0$ for $t>t^*$.

Then in the interval $[t^*, +\infty)$ equation (1) can be written in the following form: $u''+a(t)u=0$, where $a(t)=t^{-2}[\phi(t, v(t))]/v(t)$.

According to (4) and (11),

$$a(t) \geqq \frac{\phi(t_0, v(t))}{v(t)}\, t^{-2} \geqq \frac{1+\gamma}{4}\, t^{-2} \quad \text{for } t \geqq t^*.$$

Thus, according to Kneser's theorem, $u(t)$ is an oscillatory function. The contradiction thus obtained proves the theorem.

COROLLARY ([**1**]). *If* $0<\gamma<1$, $q(t)t^{(\gamma+3)/2}>0$ *and* $(q(t)t^{(\gamma+3)/2})d/dt\geqq 0$, *then every solution* $u(t)$ *of* (1') *such that* $u(t_0)=0$ *and* $|u'(t_0)|$ *is sufficiently small is oscillatory.*

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TENNESSEE, KNOXVILLE, TENNESSEE 37916

INSTITUTE OF APPLIED MATHEMATICS, TBILISI STATE UNIVERSITY, TBILISI, U.S.S.R.