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A HYPONORMAL OPERATOR WHOSE SPECTRUM IS NOT A SPECTRAL SET

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ABSTRACT. Clancey has given an example of a hyponormal, nonnormal operator whose spectrum is thin and hence not a spectral set. In this note, using fairly simple techniques, we give an example of a hyponormal operator whose spectrum contains a disc and is not a spectral set.

1. **Introduction.** An operator T on a Hilbert space $\mathscr{H}$ is called hyponormal if T^*T-TT^* is a positive operator. An interesting subclass of hyponormal operators is the class of subnormals: an operator T on $\mathscr{H}$ is called subnormal if T is the restriction of a normal operator acting on a Hilbert space $\mathscr{K}\supset\mathscr{H}$. A compact subset X of the complex plane, containing the spectrum of T (denoted by $\sigma(T)$), is called a spectral set for T if $\|f(T)\|\leqq\|f\|_X$ for all rational functions f with poles off X. It is well known that the spectrum of a subnormal operator is a spectral set (see Lebow [**4**], Berberian [**1**]). Clancey [**2**] showed the existence of hyponormal operators whose spectrum is thin (in the sense of von Neumann) and hence is not a spectral set.

In this note, using fairly simple techniques, we give an example of a hyponormal operator whose spectrum is the union of a closed disc and an annulus and is not a spectral set.

2. **Preliminaries.** If $\sigma(T)=\sigma_1\cup\sigma_2$ where σ_1 and σ_2 are closed, nonempty and disjoint sets, then it is well known (see Riesz and Sz.-Nagy [**5**, p. 421]) that $\mathscr{H}=\mathscr{H}_1\dotplus\mathscr{H}_2$ where $\mathscr{H}_1$ and $\mathscr{H}_2$ are invariant under T and $\sigma(T|\mathscr{H}_i)=\sigma_i$, $i=1,2$. Also the subspaces $\mathscr{H}_1$ and $\mathscr{H}_2$ have the property: if $\mathscr{M}$ is any subspace of $\mathscr{H}$ invariant under T such that $\sigma(T|\mathscr{M})\subset\sigma_i$ then $\mathscr{M}\subset\mathscr{H}_i$, $i=1$ or 2 (see Colojoara and Foias [**3**, p. 26]). We shall refer to this result as the Riesz Decomposition Theorem. Furthermore, it was observed by Lebow [**4**] and Williams [**6**] that if in the above case we assume that $\sigma(T)$ is a spectral set for T, then $\mathscr{H}=\mathscr{H}_1\oplus\mathscr{H}_2$.

3. **The example.** Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two subspaces of a Hilbert space $\mathscr{K}$ with the property that there are orthonormal bases $\{e_i\}_{i=-\infty}^{\infty}$ and

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$\{f_j\}_{j=1}^{\infty}$ of $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively such that $(f_j, e_i)=0$ for all i and j except when $i=0$ and $j=1$. Let $(f_1, e_0)=\theta$ and suppose that $0<|\theta|^2<1/32$.

Let $V:\mathscr{H}_1\to\mathscr{H}_1$;

$$Ve_i = 4e_{i+1}, \quad i<0, \quad Ve_0 = 8e_1, \quad \text{and}$$
$$Ve_i = 20e_{i+1}, \quad i \geqq 1.$$

Let $U:\mathscr{H}_2\to\mathscr{H}_2$;

$$Uf_1 = f_2, \quad \text{and} \quad Uf_i = 3f_{i+1}, \quad i \geqq 2.$$

First of all we remark that $\mathscr{H}=\mathscr{H}_1\dotplus\mathscr{H}_2$ is a closed subspace of $\mathscr{K}$. Thus $T=V\dotplus U$ is a bounded operator on $\mathscr{H}$; in other words $T=\begin{pmatrix} V & 0\\ 0 & U\end{pmatrix}$ on $\mathscr{H}_1\dotplus\mathscr{H}_2$. Simple computations show that

$$\begin{aligned}
T^*f_1 &= 4\theta e_{-1}, & T^*e_0 &= 4e_{-1},\\
T^*f_2 &= \rho f_1 - \theta\rho e_0, & T^*e_1 &= 8\rho e_0 - 8\bar{\theta}\rho f_1,\\
T^*f_i &= 3f_{i-1} & &\text{for } i \geqq 3,\\
T^*e_i &= 20e_{i-1} & &\text{for } i \geqq 2, \text{ and}\\
T^*e_i &= 4e_{i-1} & &\text{for } i \leqq -1, \text{ where } \rho = (1-|\theta|^2)^{-1}.
\end{aligned}$$

Let P denote the orthogonal projection of $\mathscr{H}$ onto the span of f_1, f_2, e_0 and e_1. It is quite easy to see that $\|T^*(I-P)x\|\leqq\|T(I-P)x\|$ for any $x\in\mathscr{H}$ and $(I-P)T^*TP=PTT^*(I-P)=0$. Thus in order to show that T is a hyponormal operator, it is enough to prove that $\|T^*x\|\leqq\|Tx\|$ for all x in the span of f_1, f_2, e_0 and e_1.

For any $\alpha, \beta, \gamma, \delta \in \boldsymbol{C}$,

$$\begin{aligned}
&\|T^*(\alpha f_1 + \beta e_0 + \gamma f_2 + \delta e_1)\|^2\\
&\qquad = \|4(\alpha\theta + \beta)e_{-1} + (\gamma\rho - 8\delta\bar{\theta}\rho)f_1 + (8\delta\rho - \gamma\theta\rho)e_0\|^2\\
&\qquad = 16\,|\alpha\theta + \beta|^2 + \rho^2\,\|(\gamma - 8\delta\bar{\theta})f_1 + (8\delta - \gamma\theta)e_0\|^2\\
&\qquad \leqq 32\{|\alpha|^2\,|\theta|^2 + |\beta|^2\} + 4(1+|\theta|^2)\rho^2[|\gamma|^2 + 64\,|\delta|^2].
\end{aligned}$$

On the other hand

$$\begin{aligned}
\|T(\alpha f_1 + \beta e_0 + \gamma f_2 + \delta e_1)\|^2 &= \|\alpha f_2 + 8\beta e_1 + 3\gamma f_3 + 20\delta e_2\|^2\\
&= |\alpha|^2 + 64\,|\beta|^2 + 9\,|\gamma|^2 + 400\,|\delta|^2.
\end{aligned}$$

Since $0<|\theta|^2<1/32$ it follows that $(1+|\theta|^2)\rho^2<5/4$ and now it is easy to see that

$$\|T^*x\|^2 \leqq \|Tx\|^2 \quad \text{for } x = \alpha f_1 + \beta e_0 + \gamma f_2 + \delta e_1.$$

Thus T is a hyponormal operator.

We now wish to show that $\sigma(T)$ is not a spectral set for T. First note that $\sigma(T|\mathscr{H}_1)=\sigma(V)=\{z\in \boldsymbol{C}: 4\leqq|z|\leqq 20\}=\sigma_1$ and $\sigma(T|\mathscr{H}_2)=\sigma(U)=\{z\in \boldsymbol{C}: |z|\leqq 3\}=\sigma_2$. Thus $\sigma(T)=\sigma_1\cup\sigma_2$, σ_i is closed and $\sigma_1\cap\sigma_2=\Phi$. If $\sigma(T)$ were a spectral set for T, then by the Riesz Decomposition Theorem and by the observation in Lebow [**4**], $\mathscr{H}$ would equal $\mathscr{H}'_1\oplus\mathscr{H}'_2$ where $\sigma(T|\mathscr{H}'_i)=\sigma_i$, $i=1,2$. Since $\mathscr{H}=\mathscr{H}_1\dotplus\mathscr{H}_2$ and $\sigma(T|\mathscr{H}_i)=\sigma_i$ it follows from the preliminary remarks that $\mathscr{H}_i\subset\mathscr{H}'_i$, $i=1,2$. Hence $\mathscr{H}_1=\mathscr{H}'_1$ and $\mathscr{H}_2=\mathscr{H}'_2$. Thus if $\sigma(T)$ were a spectral set for T, $\mathscr{H}_1$ would be orthogonal to $\mathscr{H}_2$. But $\mathscr{H}_1$ is not orthogonal to $\mathscr{H}_2$ and hence $\sigma(T)$ is not a spectral set for T.

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