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THE HOMOTOPY GROUPS OF SPACES WHOSE COHOMOLOGY IS A Z_p TRUNCATED POLYNOMIAL ALGEBRA

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ABSTRACT. Let K be a 1-connected CW complex with

$$H^*(K; Z_p) = Z_p[x]/x^q.$$

On such a space one can define a "Hopf" invariant homomorphism $h: \pi_{qn-1}(K) \to Z$ in two ways. We prove both definitions are equivalent and show that ${}_p\pi_i(K) \simeq {}_p\pi_{i-1}(S^{n-1}) \oplus {}_p\pi_i(S^{qn-1})$ if and only if there is an $\alpha \in \pi_{qn-1}(K)$ such that $(h(\alpha), p)=1$. As immediate corollaries of this we get a result of Toda on the homotopy groups of the reduced product spaces of spheres and a well-known result of Serre on the odd primary parts of the homotopy groups of spheres.

Let K be a simply connected CW complex such that $H^*(K, Z_p) = Z_p[x]/x^q$, $|x|=n$, p a prime. Up to p-equivalence Toda [14] has shown that we may consider $K=S^n \cup e^{2n} \cup \cdots \cup e^{(q-1)n}$ and we will restrict ourselves to such spaces. For these spaces one can define the Hopf invariant in two ways: The first is a homomorphism $h: \pi_{qn-1}(K) \to Z$, and is essentially a generalization due to James [8] of Steenrod's [12] definition of the Hopf invariant, and the second, which we will show generalizes the first, is a homomorphism $H: \pi_i(K) \to \pi_{i+1}(\Sigma^{(q-1)n}K)$ for all i and is Toda's [15] generalization of G. W. Whitehead's Hopf invariant [16].

We will investigate these Hopf invariants and prove the following.

THEOREM 0.1. *${}_p\pi_i(K) \simeq {}_p\pi_{i-1}(S^{n-1}) \oplus {}_p\pi_i(S^{qn-1})$ if and only if there exists $\alpha \in \pi_{qn-1}(K)$ such that $(h(\alpha), p)=1$. (${}_pG$ is the group $G/\bar{G}$ where $\bar{G} \subset G$ is the subgroup of torsion elements whose orders are prime to p. ${}_pG$ will be called the p-component.)*

By taking $K=S^n_q$, James' qth reduced product space of S^n [6], we get the following results as corollaries.

COROLLARY 0.2. *If $q<p$, ${}_p\pi_i(S^{2m}_{q-1}) \simeq {}_p\pi_{i-1}(S^{2m-1}) \oplus {}_p\pi_i(S^{2mq-1})$.*

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This is just 4.4 and 4.5 of Toda [15]. We will also show that this result fails for $q=p$, $m>1$.

By taking $K=S^{2m}$ we get as a corollary the well-known result of Serre [9].

COROLLARY 0.3. ${}_p\pi_i(S^{2m})\simeq {}_p\pi_{i-1}(S^{2m-1}) \oplus {}_p\pi_i(S^{4m-1})$ *for all odd primes* p.

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1. **Hopf invariants.** Consider $L=S^{n_1}\cup e^{n_2}\cup\cdots\cup e^{n_{r-1}}$ with $n_{i+1}-1>n_i\geqq 2$ and let $n_r=n_1+n_{r-1}$. Given $\beta\in\pi_{n_r-1}(L)$ form $K=L\cup_\beta e^{n_r}$ and let x_i be a generator in dimension n_i of $H^*(K)$.

DEFINITION 1.1. The Hopf invariant of β, $h(\beta)$, is that integer m such that $x_1x_{n_{r-1}}=mx_{n_r}$.

James [8] has shown that this defines a homomorphism $h:\pi_{n_r-1}(L)\to \boldsymbol{Z}$. Note that if $r=2$ then $h(\beta)$ is the usual Hopf invariant as defined by Steenrod [12].

Let $\beta\in\pi_i(L)$ and let $g:L\to L\vee S^{n_{r-1}}$ be such that:

(i) $g|(L)^{n_{r-2}}=1$, $((L)^{n_{r-2}}$ is the n_{r-2} skeleton of $L)$,

(ii) $L\to^g L\vee S^{n_{r-1}}\to^v L$, where v is the retraction map, is homotopic to the identity, and

(iii) g maps $\bar{e}^{n_{r-1}}\subset e^{n_{r-1}}$ onto $S^{n_{r-1}}$ with degree one, and $\bar{e}^{n_{r-1}}=g^{-1}(S^{n_{r-1}})$.

Since

$$\pi_i(X\vee Y)\simeq \pi_i(X)\oplus\pi_i(Y)\oplus\pi_{i+1}(X\times Y, X\vee Y),$$

let $p:\pi_i(L\vee S^{n_{r-1}})\to\pi_{i+1}(L\times S^{n_{r-1}}, L\vee S^{n_{r-1}})$ be the projection and let $q:(L\times S^{n_{r-1}}, L\vee S^{n_{r-1}})\to(\Sigma^{n_{r-1}}L, *)$ be the collapsing map.

DEFINITION 1.2. $H(\beta)$, the Hopf invariant of $\beta\in\pi_i(L)$, is $q_*pg_*(\beta)$.

It should be noted that this definition is a direct generalization of G. W. Whitehead's Hopf invariant [16] and appears in [15].

The remainder of this section will be devoted to proving

PROPOSITION 1.3. *If* $\beta\in\pi_{n_r-1}(L)$ *then there exists a generator* $\delta\in\pi_{n_r}(\Sigma^{n_{r-1}}L)$ *such that* $H(\beta)=h(\beta)\delta$.

James [8] in studying sphere bundles over spheres proved

THEOREM. *Let* $L=S^q\cup_\alpha e^n$ *and* $i_q\in\pi_q(S^q)$ *be a generator. Then there exists* $\beta\in\pi_{n+q-1}(L)$ *with* $h(\beta)=m$ *if and only if the Whitehead product* $m[\alpha, i_q]=\alpha_*\tau$, *for some* $\tau\in\pi_{n+q-2}(S^{n-1})$.

This theorem and the lemmas preceding it all generalize immediately to complexes L as defined above. We will therefore only state the results we need. I would like to thank Professor Porter for communicating this fact to me.

Let $N=(L)^{n_r-2}$ and $\alpha \in \pi_{n_{r-1}-1}(N)$ be such that $L=N \cup_\alpha e^{n_r-1}$. Let $i_n \in \pi_{n_1}(N)$ be a representative of the inclusion map $i_n: S^{n_1} \hookrightarrow N$.

THEOREM 1.4. *There exists $\beta \in \pi_{n_r-1}(L)$ such that $h(\beta)=m$ if and only if $m[\alpha, i_n]$ is in the image of $\alpha_*: \pi_{n_r-2}(S^{n_{r-1}-1}) \to \pi_{n_r-2}(N)$.*

PROOF (JAMES [8, p. 377]).

Let $\sigma \in \pi_{n_{r-1}}(L, N)$ be such that $\partial(\sigma)=\alpha$ (i.e. the class of the attaching map α), let $i_*: \pi_i(L) \to \pi_i(L, N)$ in the homotopy sequence of the pair L, N, and let $[\sigma, i_n]$ be the relative Whitehead product defined by Blakers and Massey [2].

THEOREM 1.5 (JAMES [8, p. 377]). *Let $\beta \in \pi_{n_r-1}(L)$. Then $i_*(\beta)= m[\sigma, i_n]+\sigma \circ \rho$ where $h(\beta)=m$ and $\rho \in \pi_{n_r-1}(E^{n_r-1}, S^{n_{r-1}-1})$.*

Finally let $M=S^{n_r-1} \vee L$ with $\kappa_i \in \pi_i(M)$ representing $S^i \to M$, $i=n_i$, n_{r-1} and let $u: L \hookrightarrow M$ be the inclusion.

THEOREM 1.6 (JAMES [8, p. 380]). *Let $\beta \in \pi_{n_r-1}(L)$ with $h(\beta)=m$. Then $g_*(\beta)=u_*(\beta)+m[\kappa_{n_{r-1}}, \kappa_{n_1}]+\kappa_{n_{r-1}} \circ \rho$.*

PROOF OF 1.3. In this dimension q_* is an isomorphism and observe that $\pi_{n_r}(L \times S^{n_r-1}, L \vee S^{n_r-1})$ is generated by $p([K_{n_{r-1}}, K_{n_1}])$. Assume that $p(\beta)=m$ so that, by 1.6, $g_*(\beta)=u_*(\beta)+m[K_{n_{r-1}}, K_{n_1}]+K_{n_{r-1}} \circ \rho$. But $h_*(\beta)$ lies in the image of $\pi_{n_{r-1}}(L)$ and $K_{n_{r-1}} \circ \rho$ lies in the image of $u_{n_{r-1}}(S^{n_r-1})$ in the direct sum decomposition of $\pi_{n_r-1}(L \vee S^{n_r-1})$ so that $\pi(u_*(\beta))=p(K_{n_{r-1}} \circ \rho)=0$. Therefore

$$pg_*(\beta) = mp([K_{n_{r-1}}, K_{n_1}]) \quad \text{or} \quad q_*pg_*(\beta) = m\delta$$

where δ is an appropriate generator of $\pi_{n_r}(\Sigma^{n_r-1}L)$.

2. **Proof of 0.1.** Let p be an odd prime, $H^*(K; Z_p) \simeq Z_p[x]/x^q$, $|x|=n$ even.

Assume $\alpha \in \pi_{qn-1}(K)$ with $(h(\alpha), p)=1$. Let α represent $\alpha: S^{qn-1} \to K$ and Z_α the mapping cylinder of α. Replace the inclusion $S^{qn-1} \to Z_\alpha$ with a fibration $F \xrightarrow{i} E \xrightarrow{p} Z_\alpha$ and let (E_r, d_r) be the (mod p) cohomology Serre spectral sequence for this fibration.

LEMMA 2.1.

$$\begin{aligned} E_2^{0,l} &= Z_p, && l = n-1, \\ &= 0, && \textit{otherwise}, \ 0 < l < q^{n-1}. \end{aligned}$$

PROOF. Since $H^*(E; Z_p) \simeq H^*(S^{qn-1}; Z_p)$ we must have $Z_p = E_2^{0,n-1} \simeq H^{n-1}(F; Z_p)$. Let y be the generator of $E_2^{0,n-1}$ such that $d_r y = 0$, $r < n$, $d_n y = x$.

Also since $H^*(Z_\alpha; Z_p) \simeq Z_p[x]/x^q = E_2^{*,0}$ we get $d_n(y \otimes x^m) = x^{m+1}$ for $m < q-1$.

LEMMA 2.2. *F is a Z_p-cohomology $n-1$ sphere.*

All that we must show given 2.1 is that $E_2^{(q-1)n,n-1} = E_\infty^{(q-1)n,n-1}$ or equivalently that $E_2^{0,qn-1} = 0$.

Let $(\bar{E}_r, \bar{d}_r)$ be the Serre spectral sequence for $F \to^i (E, S) \to^p (Z_\alpha, S^{qn-1})$ where $S = p^{-1}(S^{qn-1})$.

Then $H^*(Z_\alpha, S^{qn-1}; Z_p) \simeq H^*(K \cup_\alpha e^{qn}; Z_p)$ and, as we have assumed $h(\alpha, p) = 1$, we get $H^*(K \cup_\alpha e^{qn}; Z_p) \simeq Z_p[x]/x^{q+1}$. As in 2.1 we have $y \in \bar{E}_2^{0,n-1}$ such that $\bar{d}_n y = x$ but now $d_n(y \otimes x^m) = x^{m+1}$, $m < q$, which implies $E_2^{0,qn-1} = 0$.

LEMMA 2.3. *If there is $\alpha \in \pi_{qn-1}(K)$ such that $(h(\alpha), p) = 1$, then*

$$_p\pi_i(K) \simeq {}_p\pi_i(S^{qn-1}) \oplus {}_p\pi_{i-1}(S^{n-1}).$$

PROOF. Since F is a cohomology $n-1$ sphere we have $f: S^{n-1} \to F$ induces an isomorphism in Z_p homology and, therefore, an isomorphism $f_*: {}_p\pi_i(S^{n-1}) \to {}_p\pi_i(F)$. But $S^{n-1} \to^f F \to S^{qn-1}$ is trivial and from the homotopy exact sequence we get a splitting and hence

$$_p\pi_i(K) \simeq {}_p\pi_i(S^{qn-1}) \oplus {}_p\pi_{i-1}(S^{n-1}).$$

PROOF OF 0.1. Assume $h(\alpha) = kp$ for all $\alpha \in \pi_{nq-1}(K)$ and that

$$_p\pi_i(K) \simeq {}_p\pi_i(S^{qn-1}) \oplus {}_p\pi_{i-1}(S^{n-1}).$$

Let $L = (K)^{(q-2)n}$ with $K = L \cup_\beta e^{(q-1)n}$. Since $H^*(K; Z_p) = Z_p[x]/x^q$ we have $(h(\beta), p) = 1$ and we have, by 2.3,

$$_p\pi_i(L) \simeq {}_p\pi_i(S^{(q-1)n-1}) \oplus {}_p\pi_{i-1}(S^{n-1}).$$

Further $i: L \to K$ induces $i_*: {}_p\pi_i(L) \to {}_p\pi_i(K)$, one can easily see that $i_*|_p\pi_{i-1}(S^{n-1})$ is an isomorphism and $i_*|_p\pi_i(S^{(q-1)n-1})$ is trivial.

Look at the following section of the homotopy sequence for (K, L):

$$\begin{aligned}(2.4) \qquad \cdots \longrightarrow {}_p\pi_{qn-1}(L) \xrightarrow{i_*} {}_p\pi_{qn-1}(K) \xrightarrow{j_*} {}_p\pi_{qn-1}(K, L) \\ \xrightarrow{\partial} {}_p\pi_{qn-2}(L) \xrightarrow{i_*} \pi_{qn-2}(K) \longrightarrow \cdots.\end{aligned}$$

James [6] has shown that $\pi_{qn-1}(K, L) \simeq Z \oplus \pi_{qn-2}(S^{(q-1)n-1})$ and, by 1.5,

$j_*(\alpha)$ is $h(\alpha)$ times the generator of Z. Thus 2.4 yields

$$\begin{array}{ccccc}
{}_p\pi_{qn-2}(S^{n-1}) & \xrightarrow{\simeq} & {}_p\pi_{qn-2}(S^{n-1}) & \xrightarrow{0} & {}_p\pi_{qn-2}(S^{(q-1)n-1}) \\
\oplus & & \oplus & & \oplus \\
{}_p\pi_{qn-1}(S^{(q-1)n-1}) & \xrightarrow{0} & {}_p\pi_{qn-1}(S^{qn-1}) & \xrightarrow{u} & Z
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & {}_p\pi_{qn-2}(S^{(q-1)n-1} & \xrightarrow{0} & {}_p\pi_{qn-2}(S^{qn-1}) \\
& \oplus & & \oplus \\
\longrightarrow & {}_p\pi_{qn-3}(S^{n-1}) & \xrightarrow{\simeq} & {}_p\pi_{qn-3}(S^{n-1})
\end{array}$$

but u is multiplication by $h(\alpha)=kp$ and we must have ${}_p\pi_{qn-2}(L)\simeq {}_p\pi_{qn-1}(S^{(q-1)n-1}) \oplus {}_p\pi_{qn-3}(S^{n-1}) \oplus_p Z_{kp}$ which is a contradiction.

3. **Appendix.** Corollaries 0.1 and 0.2 are easily proved by noting that $H^*(S_q^{2m}; Z_p)\simeq Z_p[x]/x^{q+1}$ if $q<p$.

To show that 0.2 fails for $q=p$ by 0.1 it is sufficient to show that

PROPOSITION 3.1. *For $m>1$ there is no element $\alpha \in \pi_{2mp-1}(S_{p-1}^{2m})$ with Hopf invariant prime to p.*

PROOF. Assume there is an $\alpha \in \pi_{2mp-1}(S_{p-1}^{2m})$ with $(h(\alpha), p)=1$. Since $H^*(S_{p-1}^{2m}; Z_p)=Z_p[x]/x^p$ we have $H^*(S_{p-1}^{2m} \cup_\alpha e^{2mp}; Z_p)=Z_p[x]/x^{p+1}$. But Hardie [3, p. 248] has shown that this is equivalent to a nontrivial mod p Hopf invariant

$$P^m: H^{2m+1}(S^{2m+1} \cup e^{2mp+1}; Z_p) \to H^{2mp+1}(S^{2m+1} \cup e^{2mp+1}; Z_p)$$

which does not exist [1, Theorem D].

S_q^n is obtained from S_{q-1}^n by $S_q^n=S_{q-1}^n \cup e^{qn}$; call the attaching map $[i]^m$ (see [3]).

It is easy to see from the homotopy exact sequence for the pair $(S_\infty^{2m}, S_{p-1}^{2m})$ that $\pi_{2mp-1}(S_{p-1}^{2m})\simeq Z\oplus$Torsion and we get

COROLLARY 3.2. *$[i]^p \in \pi_{2mp-1}(S_{p-1}^{2m})$, $m>1$ generates an infinite cyclic summand.*

Let τ be the generator of the infinite cyclic summand of $\pi_{2mp-1}(S_{p-1}^{2m})$ such that $[i]^p=a\tau$, $a \in Z^+$. Assume $h(\tau)=b$ then $h([i]^p)=h(a\tau)=ab$.

But $H^*(\Omega S^{2m+1})=P_D[x]$ and $H^i(\Omega S^{2m+1})\simeq H^i(S_p^{2m})$, $i\leqq 2m(p+1)-1$. Since $S_p^{2m}=S_{p-1}^{2m} \cup_{[i]} pe^{2mp}$ we have $h[i]^p=p$ or $ab=p$. By 3.1 $b\neq 1$. Thus $b=p$ and $a=1$.

Using 3.2 we get a result due to Hardie [3].

COROLLARY 3.3. *$[i, [i]^{p-1}]$ is of order p in $\pi_{2mp-2}(S_{p-2}^{2m})$, $m>1$.*

PROOF. By [4] we have $p[i, [i]^{p-1}]=0$ and if $[i, [i]^{p-1}]=0$ by 1.4 there exists $\beta \in S^{2m}_{p-1}$ with $(h(\beta), p)=1$ which contradicts 3.1.

We also get as a corollary the following well-known result due to Hilton and Whitehead [5].

COROLLARY 3.4. *Let $\nu \in \pi_7(S^4)$ be the attaching map of the quaternionic projective plane (i.e., $HP^2=S^4 \cup_\nu e^8$). Then $2\nu \neq [i, i]$.*

PROOF. We have the following commutative diagram of cofibrations

$$\begin{tikzcd} S^7 \arrow[r, "2\nu"] \arrow[d, "2"'] & S^4 \arrow[r] \arrow[d, "1"] & S^4 \cup_{2\nu} e^8 \arrow[d, "f"] \\ S^7 \arrow[r, "\nu"] & S^4 \arrow[r] & HP^2 \end{tikzcd}$$

By the five lemma we have $H_*(S^4 \cup_{2\nu} e^8; Z_3) \xrightarrow{\sim} H_*(HP^2; Z_3)$ and hence get an isomorphism on the three component ${}_3\pi_*(S^4 \cup_{2\nu} e^8) \simeq {}_3\pi_*(HP^2)$. Since $H^*(HP^2; Z_3)=Z_3[x]/x^4$ we have by 0.1:

$${}_3\pi_i(HP^2) \simeq {}_3\pi_{i-1}(S^2) \oplus {}_3\pi_i(S_{11}) \quad \text{and therefore}$$

$${}_3\pi_i(S^4 \cup_{2\nu} e^8) \simeq {}_3\pi_{i-1}(S^3) + {}_3\pi_i(S^{11}).$$

Thus if $2\nu=[i, i]$ there exists $\alpha \in \pi_{11}(S^4_2)$ such that $(h(\alpha), 3)=1$ but this contradicts 3.2.

It should be noted that in considering 0.1 the only examples of spaces we have given have been reduced product spaces for spheres and for these we get nontrivial Hopf invariants if and only if $q<p$. In fact, Adams and Atiyah [1] have shown there exists no complex K with $H^*(K; Z_p)=Z_p[x]/x^{p+1}$ for p an odd prime, $|x|=n$ if $n/2$ does not divide $p-1$. So that in these cases there can exist no element in $\pi_{pn-1}(K^{(p-1)n})$ of Hopf invariant prime to p. It is reasonable to ask then if 0.1 is any more general than Corollary 0.2.

But D. Sullivan [13] has shown that S^{2n+1} is a loop space mod p if $m/p-1$. Therefore if we localize, in the sense of Sullivan at the prime p, $(S^{2m-1})_p$ has a classifying space $(BS^{2m-1})_p$ and $H^*(BS^{n-1}; Z_p)=Z_p[x]$.

It is perhaps interesting to note that although in this case $(S^{2m-1})_p$ has and A_∞ structure (see [11]), by 3.1, $(S^n_{p-1})_p$ yields an A_{p-1} form which cannot be extended to an A_p form.

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