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REALCOMPACTIFICATIONS OF PRODUCTS OF ORDERED SPACES

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ABSTRACT. The equality $v(X\times Y)=vX\times vY$ is studied for the case when one of the factors is a linearly ordered topological space (LOTS). Among the results obtained are the following:

1. If X is any separable realcompact space and Y is any LOTS of nonmeasurable cardinal, then $v(X\times Y)=vX\times vY$.
2. If X is a nonparacompact LOTS, then there is a paracompact LOTS Y such that $v(X\times Y)\neq vX\times vY$.
3. For any pair X, Y of well-ordered spaces, $v(X\times Y)=vX\times vY$.

The circumstance in which the Hewitt realcompactification $v(X\times Y)$ of the product $X\times Y$ and the product $vX\times vY$ of the respective Hewitt realcompactifications of the factors are equivalent extensions of the space $X\times Y$ has been studied by many researchers in a variety of contexts. The question of the equivalence of these two extensions was first considered by Hewitt in his fundamental paper [**11**]. Since then, the problem has been attacked from the points of view of general topology ([**1**], [**2**], [**3**], [**8**], [**9**], [**17**]), uniform space theory ([**10**], [**12**]), function space theory ([**13**], [**14**], [**15**]), and hybrid theories [**16**]. A complete understanding of the properties of the spaces X, Y, and $X\times Y$ that essentially influence the equality $v(X\times Y)=vX\times vY$ has thus far proved to be elusive. The rather esoteric notion of a measurable cardinal [**7**, p. 161] appears with annoying regularity in the theory. As we indicate in §2 below, cardinalities of the spaces in question can be essential to the validity of the equality $v(X\times Y)=vX\times vY$ apart from the measurability pathology. Spaces in which the continuous real-valued functions are completely determined by the compact subsets (k'-spaces) seem to play an important role ([**2**], [**13**], [**15**]). We suspect that the k'-spaces are relevant to the problem simply because these are the spaces which have nicely behaving real-valued function spaces equipped with the compact-open topology. In fact, one can find many of the known theorems for $v(X\times Y)=vX\times vY$ implicit in the chapter on function spaces in [**4**].

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In the present paper, we retreat from the general context to consider the equality $v(X\times Y)=vX\times vY$ in the more restrictive situation when one of the spaces X or Y is a linearly ordered topological space (hereinafter denoted as "LOTS"). Among the results obtained is the fact that in the category of well-ordered topological spaces (such a space will hereinafter be denoted as "WOTS") $v(X\times Y)=vX\times vY$ holds universally with no restriction on cardinality. Further, it is established that if X is a separable realcompact space ("space" will always mean completely regular Hausdorff space in the sequel) and Y is any LOTS of nonmeasurable cardinal, then $v(X\times Y)=vX\times vY$. It is finally determined that if X is a nonparacompact LOTS, then there is a paracompact LOTS Y such that card $X=$ card Y and $v(X\times Y)\neq vX\times vY$.

1. **Terminology and notation.** Our basic references for notation are [**6**] and [**7**]. For the convenience of the reader, we summarize below the pertinent parts of [**6**] and other definitions and results which will be referred to in the sequel.

1.1. For a space X, vX denotes the Hewitt realcompactification of X [**7**, pp. 116–119]. The symbolism $v(X\times Y)=vX\times vY$ means that the space $X\times Y$ is C-embedded in the space $vX\times vY$.

1.2. For an ordinal σ, $W(\sigma)$ denotes the ordinal space of all ordinals less than σ equipped with the order topology.

1.3. The smallest ordinal of cardinality $\aleph_\alpha$ is denoted by ω_α; we will take the point of view that the only distinction between $\aleph_\alpha$ and ω_α is one of notation.

1.4. Let L be a LOTS. A *gap* of L is defined as a Dedekind cut $(A|B)$ such that A has no last element and B no first. If $A=\varnothing$ or $B=\varnothing$, $(A|B)$ is called an *end-gap*. The linearly ordered set consisting of L together with all gaps of L is denoted by L^+.

1.5. Let L be a LOTS and let $u=(A|B)$ be a gap of L. If u is not a left end-gap, then the *left character* of u is the least ordinal λ which is cofinal with A. If u is not a right end-gap, then the *right character* of u is the least ordinal ρ such that ρ^* ("*" denotes reverse order) is coinitial with B.

1.6. (i) Let L be a LOTS and let ω_α be any regular initial ordinal. An increasing or decreasing sequence $s=\{x_\sigma\}_{\sigma<\omega_\alpha}$ of points of L^+ is a *Q-sequence* if for every nonzero limit ordinal $\lambda<\omega_\alpha$, the limit in L^+ of the segment $\{x_\sigma\}_{\sigma<\lambda}$ of s is a gap of L.

(ii) If s is a Q-sequence and if the gap u is the limit of the entire sequence s, s is called a *Q-sequence* at u.

(iii) A gap u is called a *Q-gap from the left* (*right*) if there exists an increasing (decreasing) Q-sequence at u.

(iv) A gap u is called a *Q-gap* if it is a Q-gap from the left and from the right (or the appropriate one if u is an end-gap).

1.7. Let X be a LOTS. For each gap u of X add elements l_u and r_u (or only the appropriate one if u is an end-gap) to form the LOTS X'' ordered in the natural way ($l_u < r_u$) preserving the order of X. Form X' from X'' by deleting every element l_u for which the gap u is a Q-gap from the left, and every r_u for which u is a Q-gap from the right. Then, X is a dense C-embedded subspace of X' and hence $X \subset X' \subset vX$. If card X is nonmeasurable, then $X' = vX$ (see [**6**, pp. 359–360]).

1.8. (Gilman and Henriksen). A LOTS X is paracompact if and only if every gap of X is a Q-gap. A LOTS X is realcompact if and only if every gap of X is a nonmeasurable Q-gap.

1.9. (Glicksberg). If $X \times Y$ is pseudocompact, then $v(X \times Y) = vX \times vY$.

1.10. (Comfort). If X is locally compact, realcompact of nonmeasurable cardinal, then $v(X \times Y) = vX \times vY$ for every space Y.

1.11. (Hager). If X and Y are spaces such that $X = \bigcup_n X_n$, $Y = \bigcup_n Y_n$, X_n is completely separated from $X - X_{n+1}$ and Y_n is completely separated from $Y - Y_{n+1}$ for each n, and $X_n \times Y_n$ is pseudocompact for each n, then $v(X \times Y) = vX \times vY$.

2. **Linearly ordered spaces and** $v(X \times Y)$. It is established in [**6**] that a gap u of a LOTS X is not a Q-gap from the left (right) if and only if every real-valued continuous function on X is constant on a left (right) interval at u. The following theorem indicates that the "size" of the factors can in some cases determine the validity of the equality $v(X \times Y) = vX \times vY$.

THEOREM 2.1. *Suppose X is a nonparacompact LOTS of nonmeasurable cardinal. Let $\{u_\lambda\}_{\lambda \in \Lambda}$ be the family of all gaps of X which are not Q-gaps from the left and $\{u_\rho\}_{\rho \in P}$ be the family of all gaps of X which are not Q-gaps from the right. Let $\{\omega_\alpha\}_{\alpha \in A}$ be the collection of left characters of $\{u_\lambda\}_{\lambda \in \Lambda}$ and $\{\omega_\beta\}_{\beta \in B}$ be the collection of right characters of $\{u_\rho\}_{\rho \in P}$. Let $\omega_\gamma = \inf\{\omega_\xi : \xi \in A \cup B\}$. Let Y be a realcompact space which has a dense subspace D such that* card $D < \omega_\gamma$. *Then $v(Y \times X) = vY \times vX$.*

PROOF. Let f be a continuous real-valued function on $Y \times X$. We wish to extend f continuously to $Y \times X'$. For each point y of Y, $f|_{\{y\} \times X'}$ can be extended continuously to the function $\hat{f}_y$ from $\{y\} \times X'$ to the real line. Define $\hat{f}(y, p) = \hat{f}_y(p)$ for y in Y and p in X'. To show that $\hat{f}$ is the desired extension of f, it suffices to show that, for each point p of $X' - X$, $\hat{f}$ is continuous on the space $Y \times (X \cup \{p\})$. Let p be a point of $X' - X$. We will assume that $p = l_{u_\lambda}$ for some $\lambda \in \Lambda$, the other case being similar. For each point d of D, there is a real number η_d and a point x_d of X such that $\hat{f}_d$ is constantly equal to η_d on $\{d\} \times [x_d, l_{u_\lambda}]$ (see [**6**, 10.6]). Our cardinality hypothesis on the set D ensures that we may choose a point x' of X such that $x_d < x' < l_{u_\lambda}$ for every d in D. It is easy to see that, for every point y of

Y, $\hat{f}_y$ is constant on $\{y\}\times[x', l_{u_\lambda}]$. Now, choose any point x'' of X such that $x'<x''<l_{u_\lambda}$. Let y_0 be any point of Y and $\varepsilon>0$. Then, there is a neighborhood U of y_0 and a neighborhood (x_1, x_2) of x'' with $x'\leqq x_1<x''<x_2<l_{u_\lambda}$ such that $f_*(U\times(x_1, x_2))\subset(f(y_0, x'')-\varepsilon, \quad f(y_0, x'')+\varepsilon)$. Then, $\hat{f}_*(U\times(x_1, l_{u_\lambda}])\subset(f(y_0, x'')-\varepsilon, f(y_0, x'')+\varepsilon)=(\hat{f}(y_0, l_{u_\lambda})-\varepsilon, \hat{f}(y_0, l_{u_\lambda})+\varepsilon)$. [The fact that $(x_1, l_{u_\lambda}]$ is a neighborhood of l_{u_λ} in $X\cup\{l_{u_\lambda}\}$ is evident from the construction of X'', noting that $(x_1, l_{u_\lambda}]=(x_1, r_{u_\lambda})\cap(X\cup\{l_{u_\lambda}\})$.]!!

COROLLARY 2.2. *Let Y be a separable realcompact space. Then, for every LOTS X of nonmeasurable cardinal, $v(X\times Y)=vX\times vY$.*

PROOF. If X is paracompact, then X is realcompact and the situation is trivial. If X is nonparacompact, then the result follows from 2.1 with the remark that $\omega_1\leqq\omega_\gamma$ since a gap with left (right) character ω_0 is a Q-gap from the left (right).!!

COROLLARY 2.3. *Let Y be a countable space. Then, for every LOTS Y of nonmeasurable cardinal, $v(X\times Y)=vX\times vY$.*

REMARK 2.4. It was established in **[16]** that there are spaces X and Y with X countable and Y with nonmeasurable cardinal such that $v(X\times Y)\neq vX\times vY$. In fact, it can be shown that if Y is any nonrealcompact extension of the countable discrete space, then there is a countable space X such that $v(X\times Y)\neq vX\times vY$.

THEOREM 2.5. *If X is a nonparacompact LOTS, then there is a paracompact LOTS Y such that $v(X\times Y)\neq vX\times vY$.*

PROOF. Let u be a gap of X which is not a Q-gap. We assume for definiteness that u is not a Q-gap from the left. Let $\{x_\sigma\}_{\sigma<\omega_\alpha}$ be an increasing sequence in X with limit u. Let $E=\{x_\sigma:\sigma$ is a nonlimit ordinal$<\omega_\alpha\}$ and let $F=E\times Z$ ("Z" denotes the discrete space of integers) be equipped with the lexicographic order. Finally, let $Y=F\cup\{l_u\}$ be equipped with the order topology where l_u follows all elements of F. Since F is a discrete subspace of Y, Y is paracompact. For each x_σ in E, let f_σ be a continuous function from $X\cup\{l_u\}$ to the closed interval $[0, 1]$ such that $f_\sigma(l_u)=0$ and $f_\sigma(x)=1$ for every $x\leqq x_\sigma$ and every $x>l_u$. [We can do this since $\{x: x\leqq x_\sigma\}\cup\{x:x>l_u\}$ is closed.] Define the real-valued function f on $X\times Y$ by the rule:

$$\begin{aligned} f(x, y) &= 1, && \text{if } y = l_u,\\ &= f_\sigma(x), && \text{if } y = (x_\sigma, z). \end{aligned}$$

It is not hard to see that f is continuous on $X\times Y$. We show that f cannot be continuously extended to the point (l_u, l_u) of $vX\times vY$. Let U be a neighborhood of (l_u, l_u) in $vX\times vY$. Choose a point x_σ of E such that

(x, y) is in U whenever $x_\sigma < x < l_u$ and $(x_\sigma, 0) < y$. Choose a point x of X such that $x_\sigma < x < l_u$ and $f_\sigma(x) < \frac{1}{2}$. Then, (x, l_u) and $(x, (x_\sigma, 1))$ are both points of U and $|f(x, 1_u) - f(x, (x_\sigma, 1))| = |1 - f_\sigma(x)| > \frac{1}{2}$. Thus, f cannot be continuously extended over $vX \times vY$.!!

REMARK 2.6. If the LOTS X in 2.5 has nonmeasurable cardinal, then Y is realcompact. We may construct a space T from Y by an appropriate topological sum so that T is a paracompact LOTS, card T=card X, and $v(X \times T) \neq vX \times vT$. Thus, barring measurable cardinals, we have shown that if X is a nonrealcompact LOTS, there is a realcompact LOTS T such that card T=card X and $v(X \times T) \neq vX \times vT$.

REMARK 2.7. Theorem 2.5 also indicates that the cardinality criterion of Theorem 2.1 is critical. Notice that the cardinality of Y in 2.5 is precisely the left character of the gap u.

3. **Well ordered spaces and** $v(X \times Y)$. The situation for well-ordered spaces is simple and satisfactory. Our main result of this section is that, with no restriction on cardinality, $v(X \times Y) = vX \times vY$ always holds in the category of well-ordered topological spaces. We begin with a few preliminary results.

LEMMA 3.1. *Every paracompact WOTS is realcompact.*

PROOF. If $W(\sigma)$ is a paracompact, noncompact WOTS, then σ is an endgap of character ω_0. Thus, $W(\sigma)$ is a countable union of compact subspaces and is realcompact [7, 8.2].!!

LEMMA 3.2. *A nonparacompact WOTS is pseudocompact.*

PROOF. Let $W(\sigma)$ be a nonparacompact WOTS. Suppose f is a real-valued continuous function defined on $W(\sigma)$. Then, there is an ordinal $\tau < \sigma$ such that f is constant on the interval $[\tau, \sigma)$. But, the interval $[0, \tau]$ is compact and hence f is bounded on the set $[0, \tau] \cup [\tau, \sigma) = W(\sigma)$.!!

LEMMA 3.3. *If $W(\sigma)$ and $W(\tau)$ are pseudocompact WOTS, then $W(\sigma) \times W(\tau)$ is pseudocompact.*

PROOF. Note that every WOTS is locally compact. The result now follows from [**5**, 3.3 and 3.4].!!

We now state and prove the main result of the section.

THEOREM 3.4. *For each pair of ordinals σ and τ, $v(W(\sigma) \times W(\tau)) = v(W(\sigma)) \times v(W(\tau))$.*

PROOF. *Case* I. Both $W(\sigma)$ and $W(\tau)$ are realcompact.

There is nothing to prove in this case.

Case II. Both $W(\sigma)$ and $W(\tau)$ are nonrealcompact.

By 3.1, 3.2, and 3.3, we conclude that $W(\sigma)\times W(\tau)$ is pseudocompact. But, then by Glicksberg's theorem (see 1.9), $v(W(\sigma)\times W(\tau))=v(W(\sigma))\times v(W(\tau))$.

Case III. $W(\sigma)$ is realcompact and $W(\tau)$ is nonrealcompact.

If $W(\sigma)$ is compact, then Glicksberg's theorem implies the result. If $W(\sigma)$ is noncompact, choose an increasing sequence $x_1, x_2, \cdots$ of nonlimit ordinals whose limit is σ. Then, $W(\sigma)=\bigcup_{n=1}^{\infty} W(x_n)$, each $W(x_n)$ is completely separated from $W(\sigma)-W(x_{n+1})$, and each $W(x_n)\times W(\tau)$ is pseudocompact ($W(x_n)$ is compact and $W(\tau)$ is pseudocompact). Thus, by Hager's theorem (see 1.11), $v(W(\sigma)\times W(\tau))=v(W(\sigma))\times v(W(\tau))$.!!

COROLLARY 3.5. *For every pair of WOTS X and Y, $v(X\times Y)=vX\times vY$.*

PROOF. There are unique ordinals σ and τ such that X is homeomorphic to $W(\sigma)$ and Y is homeomorphic to $W(\tau)$.!!

REMARK 3.6. Recall that, as a consequence of Theorem 2.5, for every nonrealcompact WOTS X, there is a paracompact LOTS Y such that $v(X\times Y)\neq vX\times vY$. If X is a realcompact WOTS with nonmeasurable cardinal, then $v(X\times Y)=vX\times vY$ for every space Y by Comfort's theorem (see 1.10).

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