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# ON FIXED POINTS OF NONEXPANSIVE MAPPINGS IN NONCONVEX SETS

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ABSTRACT. Two theorems are proved concerning the existence of fixed points of nonexpansive mappings on a certain class of nonconvex sets. This work extends the author's previous work on star-shaped sets.

Suppose $S$ is a subset of a Banach space $E$, and let $F=\{f_\alpha\}_{\alpha\in S}$ be a family of functions from $[0, 1]$ into $S$, having the property that for each $\alpha \in S$ we have $f_\alpha(1)=\alpha$. Such a family $F$ is said to be *contractive* provided there exists a function $\phi:(0, 1)\to(0, 1)$ such that for all $\alpha$ and $\beta$ in $S$ and for all $t$ in $(0, 1)$ we have

$$\|f_\alpha(t) - f_\beta(t)\| \leqq \phi(t) \, \|\alpha - \beta\|.$$

Such a family $F$ is said to be *jointly continuous* provided that if $t\to t_0$ in $[0, 1]$ and $\alpha\to\alpha_0$ in $S$ then $f_\alpha(t)\to f_{\alpha_0}(t_0)$ in $S$.

THEOREM 1. *Suppose $S$ is a compact subset of a Banach space $E$, and suppose there exists a contractive, jointly continuous family $F$ of functions associated with $S$ as described above. Then any nonexpansive self-mapping $T$ of $S$ has a fixed point in $S$.*

PROOF. For each $n=1, 2, 3, \cdots$, let $k_n=n/(n+1)$, and let $T_n:S\to S$ be defined by $T_nx=f_{Tx}(k_n)$ for all $x \in S$. Since $T(S)\subset S$ and $0<k_n<1$, we have that each $T_n$ is well-defined and maps $S$ into $S$. Furthermore, for each $n$ we have, for all $x$, $y$ in $S$,

$$\|T_nx - T_ny\| = \|f_{Tx}(k_n) - f_{Ty}(k_n)\| \leqq \phi(k_n) \, \|Tx - Ty\| \leqq \phi(k_n) \, \|x - y\|,$$

so that, for each $n$, $T_n$ is a contraction mapping on $S$. As a compact (hence closed) subset of the Banach space $E$, $S$ is a complete metric space. Therefore each $T_n$ has a unique fixed point $x_n \in S$. Since $S$ is compact, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j}\to$some $x \in S$. Since $T_{n_j}x_{n_j}=x_{n_j}$ we

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have $T_{n_j}x_{n_j} \to x$. But $T$ is continuous (since nonexpansive), and so $Tx_{n_j} \to Tx$. The joint continuity now yields

$$T_{n_j}x_{n_j} = f_{Tx_{n_j}}(k_{n_j}) \to f_{Tx}(1) = Tx.$$

It follows that $Tx = x$, since $E$ is Hausdorff. Q.E.D.

A special case of the above theorem is Theorem 1 of [**1**], where $S$ is assumed to be star-shaped. With $p$ a star-center and $k_n = n/(n+1)$ we have $f_\alpha(t) = (1-t)p + t\alpha$ so that $T_n x = f_{Tx}(k_n) = (1-k_n)p + k_n Tx$. One easily checks that

$$\|f_\alpha(t) - f_\beta(t)\| \leqq t\,\|\alpha - \beta\|$$

so that we can take $\phi(t) = t$ for $0 < t < 1$; and it is a well-known fact that $f_\alpha(t) = (1-t)p + t\alpha$ is jointly continuous in $t$ and $\alpha$.

A family $F = \{f_\alpha\}_{\alpha \in S}$ of functions from $[0, 1]$ into a set $S$ will be called *jointly weakly continuous* provided that if $t \to t_0$ in $[0, 1]$ and $\alpha \rightharpoonup \alpha_0$ in $S$ then $f_\alpha(t) \rightharpoonup f_{\alpha_0}(t_0)$ in $S$ (here $\rightharpoonup$ denotes weak convergence).

THEOREM 2. *Suppose $S$ is a weakly compact subset of a Banach space $E$, and suppose there exists a contractive, jointly weakly continuous family $F$ of functions associated with $S$ as described above and before Theorem* 1. *Then any nonexpansive weakly continuous self-mapping $T$ of $S$ has a fixed point in $S$.*

PROOF. As in Theorem 1, let $k_n = n/(n+1)$ and define $T_n : S \to S$ by $T_n x = f_{Tx}(k_n)$ for all $x \in S$ and for all $n = 1, 2, 3, \cdots$. Then, as before, each $T_n$ is a contraction mapping on $S$. Since the weak topology of $E$ is Hausdorff and $S$ is weakly compact, we have that $S$ is weakly closed and therefore strongly closed. Hence $S$ is a complete metric space (with the norm topology of the Banach space $E$), and so each $T_n$ has a unique fixed point $x_n \in S$. By the Eberlein-Šmulian theorem $S$ is weakly sequentially compact. Thus there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup$ some $x \in S$. Since $T_{n_j}x_{n_j} = x_{n_j}$ we have $T_{n_j}x_{n_j} \rightharpoonup x$. Since $T$ is weakly continuous we have $Tx_{n_j} \rightharpoonup Tx$. The joint weak continuity now yields $T_{n_j}x_{n_j} = f_{Tx_{n_j}}(k_{n_j}) \rightharpoonup f_{Tx}(1) = Tx$. Since the weak topology is Hausdorff, we now get $Tx = x$. Q.E.D.

## REFERENCE

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