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THE n-GENERATOR PROPERTY FOR COMMUTATIVE RINGS[1]

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ABSTRACT. Let D be an integral domian with identity. If for some positive integer n, each finitely generated ideal of D has a basis of n elements, then the integral closure of D is a Prüfer domain. This result generalizes to the case of commutative rings with identity that contain zero divisors.

All rings considered in this paper are assumed to be commutative. If R is a ring and if n is a positive integer, then following [7], we say that R has the n-generator property if each finitely generated ideal of R has a basis of n elements. Thus Bezout domains have the 1-generator property, Dedekind domains have the 2-generator property, and the rings of finite rank n of I. S. Cohen [2] have the n-generator property. In this note, we prove (Corollary 3) that the integral closure of a domain with the n-generator property is a Prüfer domain. Then we extend this result in Corollary 5 to a class of rings with zero divisors. We begin with an easy result on generating sets for a module.

PROPOSITION 1. *Assume that R is a ring, N is a unitary R-module, and S and T are sets of generators for N. If M is a maximal ideal of R, then there is a subset S' of S such that $|S'|\leqq|T|$ and such that $N=N'+MN$, where N' is the submodule of N generated by S'.*

PROOF. The sets $\{s+MN \mid s\in S\}$ and $\{t+MN \mid t\in T\}$ span the (R/M)-vector space N/MN, and hence each of these sets contains a basis for N/MN. If $S'\subseteq S$ is such that $\{s+MN \mid s\in S'\}$ is a basis for N/MN, then it is clear that $N=N'+MN$, and $|S'|\leqq|T|$ because any two bases for N/MN have the same cardinality.

COROLLARY 1. *Let the notation be as in Proposition 1. If $|T|=n<\infty$ and if $\{M_\lambda\}_{\lambda\in\Lambda}$ is the set of maximal ideals of R, then S contains a set of generators for N of cardinality at most $n|\Lambda|$.*

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PROOF. For each λ in Λ, we choose a subset S_λ of S such that $|S_\lambda| \leqq n$ and $N = (S_\lambda) + M_\lambda N$. If $S' = \bigcup_\lambda S_\lambda$, then $|S'| \leqq n|\Lambda|$ and it is clear that $N = N' + M_\lambda N$ for each λ in Λ. Therefore $M_\lambda(N/N') = (N/N')$ for each λ in Λ. Since N/N' is finitely generated, it follows that there is an element $m_\lambda \in M_\lambda$ such that $1 - m_\lambda$ annihilates N/N' [**12**, p. 50]. Consequently, the annihilator A of N/N' is contained in no M_λ, 1 is in A, and $N = N'$. This completes the proof of Corollary 1.

Note that the bound of Corollary 1 is of interest only in the case where Λ is finite, for if N is a finitely generated R-module, for an arbitrary ring R, then each set of generators for N contains a finite set of generators for N.

COROLLARY 2. *If R is a quasi-local ring, if N is a unitary R-module, and if S and T are sets of generators for N, where T is finite with n elements, then S contains a finite set $\{s_i\}_{i=1}^k$ of generators for N, where $k \leqq n$.*

We remark that the bound given in Corollary 1 cannot, in general, be improved. For example, if R is semi-quasi-local with n maximal ideals $M_1, M_2, \cdots, M_n$ and if for $1 \leqq i \leqq n$, $m_i \in (\bigcap_{j \neq i} M_j) - M_i$, then $S = \{m_i\}_{i=1}^n$ generates R as an R-module, but no proper subset of S generates R.

THEOREM 1. *Let a and b be nonzero elements of the integral domain D with identity, and let i and j be nonnegative integers such that $i+j=n>0$. If $a^ib^j \in (a^n, \cdots, a^{i+1}b^{j-1}, a^{i-1}b^{j+1}, \cdots, b^n)$, and if J is the integral closure of D, then $\{a, b\}J$ is invertible.*

PROOF. We prove that $\{a, b\}J_M$ is principal for each maximal ideal M of J. By assumption, there is a homogeneous polynomial f in $D[X, Y]$ of degree n such that $f(a, b) = 0$ and such that the coefficient of X^iY^j in f is -1. Therefore, the equation $0 = f(a, b)/a^n = f(1, b/a)$ shows that b/a is a root of the polynomial $f(1, Y) \in D[Y]$, where the coefficient of Y^j in $f(1, Y)$ is -1. Consequently, b/a or a/b is in J_M [**17**, p. 19], and hence $\{a, b\}J_M = aJ_M$ or bJ_M. In either case, $\{a, b\}J_M$ is principal, and $\{a, b\}J$ is invertible.

We remark that in Theorem 1, the ideal $\{a, b\}D$ need not be invertible, even in the case where D is quasi-local. For example, in $D = K[[X^2]][X^3]$, where K is a field, $X^{3n} \in (X^{2n}, X^{2(n-1)}X^3, \cdots, X^2X^{3(n-1)})$ for each integer $n > 1$, but $\{X^2, X^3\}D$ is not invertible. Our proof of Theorem 1 shows that $\{a, b\}D$ is invertible if D_M is integrally closed for each maximal ideal M of D containing $\{a, b\}$.

COROLLARY 3. *If D is an integral domain with identity and if D has the n-generator property for some positive integer n, then the integral closure of D is a Prüfer domain.*

PROOF. Since each finitely generated fractional ideal of D has a basis of n elements, each overring of D has the n-generator property. Hence to prove Corollary 3, it suffices to prove that an integrally closed quasi-local domain J with the n-generator property is a valuation ring. Let $a, b \in J-\{0\}$. By assumption, $(a, b)^n$ has a basis of n elements, and Corollary 2 implies that $\{a^{n-i}b^i\}_{i=0}^n$ contains a set of n generators for $(a, b)^n$. Consequently, $a^{n-i}b^i \in (a^n, \cdots, a^{n-i+1}b^{i-1}, a^{n-i-1}b^{i+1}, \cdots, b^n)$ for some i, and the proof of Theorem 1 shows that a/b or b/a is in J. Therefore J is a valuation ring and our proof of Corollary 3 is complete.

We remark that some special cases of Theorem 1 already appear in the literature [**1**, Proposition 3.9], [**4**, Theorem 4.7], [**5**, Proposition 20.2], but our proof of Theorem 1 is not similar to the proofs of any of these special cases.[2] Moreover, E. Matlis [**16**, Lemma 2] has given a proof of our Corollary 3 for the case $n=2$. The converse of Corollary 3 fails miserably; If F is a subfield of the field K and if K/F is infinite dimensional algebraic, then $D=F+XK[[X]]$ has integral closure $K[[X]]$, a rank 1 discrete valuation ring, but D has the n-generator property for no positive integer n.

In [**2**], I. S. Cohen considers the concept of a ring of finite rank, defined as follows. If R is a commutative ring and if n is a positive integer, then *R has rank n* if each ideal of R has a basis of n elements (more generally, this concept can be extended to a module over R; see [**9**] and its bibliography). It is clear that a ring of finite rank is Noetherian. In [**2**, §4], Cohen proved that if D is a Noetherian domain with identity of finite rank, then $\dim D \leqq 1$; this conclusion is also valid for Noetherian rings with identity [**9**]. For the domain case we obtain an easy proof of this result, based on Corollary 3.

COROLLARY 4. *If D is an integral domain with identity of finite rank, then the dimension of D is at most* 1.

PROOF. If J is the integral closure of D, then J is a Prüfer domain by Corollary 3. Since J is also Noetherian, J is a Dedekind domain, and hence $\dim J = \dim D \leqq 1$.

An integrally closed domain D of finite rank is a Dedekind domain, and hence D has rank 2. What is the structure of an integrally closed ring of finite rank? To answer this question, we seek a generalization of Theorem 1.

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[2] Two papers of H. Bass, *Torsion free and projective modules*, Trans. Amer. Math. Soc. **102** (1962), 319–327, and *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1963), 8–28, also contain results on minimal generating sets for modules and ideals. The relevant parts of Bass' papers are §1, especially Proposition 1.4, of the first paper and §7, especially Lemma 7.4, of the second.

PROPOSITION 2. *Let a and b be elements of the ring R with identity. Assume that a is regular in R and that there exist nonnegative integers i and j such that $i+j=n>0$ and $a^ib^j \in (\{a^{n-k}b^k | 0 \leqq k \leqq n, k \neq j\})$. If S is the integral closure of R, then $\{a, b\}S$ is invertible.*

PROOF. We use induction on n, the case $n=1$ being obvious. For $n=2$, we have $ab \in (a^2, b^2)$ or $a^2 \in (ab, b^2)$ or $b^2 \in (a^2, ab)$. In the first case, Proposition 3.9 of [**1**] shows that $\{a, b\}S$ is invertible. If $a^2 \in (ab, b^2) \subseteq (b)$, then b is regular in R so there is no loss of generality in assuming that $b^2 \in (a^2, ab)$—say $b^2 = ra^2 + tab$, where $r, t \in R$. Then $(b/a)^2 - t(b/a) - r = 0$, and hence $b/a \in S$ so that $\{a, b\}S = aS$ is invertible. We assume that Proposition 2 is true for $n=t$, where $t \geqq 2$, and we consider the case $n=t+1$. Since $\{a, b\}S$ is invertible if and only if $\{a, b\}S_M$ is invertible for each maximal ideal M of S, we prove that $\{a, b\}S$ is invertible under the assumption that S is quasi-local and integrally closed. There is a homogeneous polynomial $f = \sum_{k=0}^{t+1} s_k X^k Y^{t+1-k}$ in $S[X, Y]$ of degree $t+1$ such that $s_i = -1$ and $f(a, b) = 0$. Therefore $0 = f(a, b)/a^{t+1} = f(1, b/a)$. If $i=0$ we conclude that b/a is integral over S so that again $b/a \in S$ and $\{a, b\}S = aS$ is invertible; we handle the case $i=t+1$ similarly—b is then regular in S. If $0<i<t+1$, then it follows from the equation $0 = \sum_{k=0}^{t+1} s_k (b/a)^{t+1-k}$ that $s_0 b/a$ is integral over S [**5**, Lemma 7.1], and hence $s_0 b/a = u$ is in S. If u is a unit of S, then $a = u^{-1}s_0 b$, and once again $\{a, b\}S$ is principal, regular, and therefore invertible. If u is a nonunit of S, then we make the substitution $au = s_0 b$ in the equation $0 = \sum_{k=0}^{t+1} s_k a^k b^{t+1-k}$, obtaining

$$0 = au \cdot b^t + s_1 ab^t + s_2 a^2 b^{t-1} + \cdots + s_{t+1} a^{t+1}.$$

Since a is regular in S, it follows that

$$0 = (u + s_1)b^t + s_2 ab^{t-1} + \cdots + s_{t+1} a^t.$$

If $i>1$, we have a reduction to the case where $n=t$, and if $i=1$, then $u+s_1 = u-1$ is a unit of S, for u is a nonunit of S. Consequently, $b^t \in (ab^{t-1}, \cdots, a^t)$, and again the induction hypothesis implies $\{a, b\}S$ is invertible. By the principle of mathematical induction, our proof is complete.

Again our proof of Proposition 2 shows that $\{a, b\}R$ is invertible in R if R_M is integrally closed for each maximal ideal M of R containing $\{a, b\}$.

A commutative ring R with identity is a *Prüfer ring* if each finitely generated regular ideal of R is invertible [**14**, Chapter 10]. In attempting to generalize Corollary 3 to the case of rings with zero divisors, we encounter new difficulties; these difficulties stem from the fact that a Prüfer ring need not have what J. Marot in [**15**] refers to as property (P)—the property that each regular ideal is generated by its set of regular elements (see [8] for an example of a Prüfer ring that does not have property (P)). But for rings with property (P)—and this class of rings includes the classes of Noetherian rings, rings with few zero divisors [3, p. 203], and additively regular rings [11]—we obtain an extension of Corollary 3.

COROLLARY 5. *If R is a ring with identity, if R has property* (P), *and if R has the n-generator property for some positive integer n, then the integral closure of R is a Prüfer ring.*

PROOF. The hypothesis we have made concerning R carries over to each overring of R, and hence we assume, without loss of generality, that R is integrally closed. We wish to prove that each finitely generated regular ideal A of R is invertible; since R has property (P), A is generated by a finite set of regular elements. Hence to prove that R is a Prüfer ring, we need only prove that if a, b are regular elements of R, then (a, b) is invertible [5, Proposition 18.2]. Let $\{M_\lambda\}$ be the set of maximal ideals of R that contain $\{a, b\}$ and for each M_λ, let $N_\lambda = R - M_\lambda$. We observe that $M_\lambda R_{N_\lambda}$ is the unique maximal regular ideal of R_{N_λ}; thus if C is a regular ideal of R not contained in M_λ, then because C is generated by its set of regular elements, there is a regular element c of C not in M_λ—that is, C meets N_λ. This means, in particular, that $M_\lambda R_{N_\lambda}$ is the unique maximal ideal of R_{N_λ} containing a or b. Because $(a, b)^n$ has a basis of n elements, Corollary 2 shows that there is a subset H of $\{a^{n-k}b^k\}_{k=0}^n$ such that $|H| \leqq n$ and such that the image of H under the canonical imbedding of R into R_{M_λ} generates the image of $(a, b)^n$ under this imbedding. Now $R_{M_\lambda} \simeq (R_{N_\lambda})_{M_\lambda R_{N_\lambda}}$ and $M_\lambda R_{N_\lambda}$ is the unique maximal ideal of R_{N_λ} containing HR_{N_λ} or $(a, b)^n R_{N_\lambda}$; consequently $HR_{N_\lambda} = (a, b)^n R_{N_\lambda}$, and

$$a^{n-i}b^i \in (a^n, \cdots, a^{n-i+1}b^{i-1}, a^{n-i-1}b^{i+1}, \cdots, b^n)R_{N_\lambda}$$

for some i. By Proposition 2, $(a, b)R_{N_\lambda}$ is invertible so that $ab \in (a^2, b^2)R_{N_\lambda}$. It follows that $ab \in \bigcap_\lambda (a^2, b^2)R_{N_\lambda}$, and since $\{M_\lambda\}$ is also the set of maximal ideals of R that contain $\{a^2, b^2\}$, $\bigcap_\lambda (a^2, b^2)R_{N_\lambda} = (a^2, b^2)$. Therefore $ab \in (a^2, b^2)$, and (a, b) is invertible by Proposition 2.

Our next result is clear; again it generalizes to rings with zero divisors some previous considerations for integral domains.

COROLLARY 6. *If R is a ring with identity of finite rank, then the integral closure of R is a Noetherian Prüfer ring.*

Davis has given a satisfactory description of Noetherian Prüfer rings in [3, §3], so we shall not pursue the subject here, but we do make a few remarks concerning such rings. A zero-dimensional semilocal ring is an integrally closed Prüfer ring of finite rank (see [9]), but the rank of such a ring need not be less than n for any fixed positive integer n. A Noetherian Prüfer ring R of finite rank need not have nilradical 0, even if its dimension is 1 and R is indecomposable, but it is true that the dimension of a Noetherian Prüfer ring is at most 1.

A few open questions lie close at hand. In connection with Corollary 5, we have the following question. If R is a ring with identity with the n-generator property, does R have property (P)? In particular, if A is an invertible ideal of R, is A generated by its set of regular elements? (The answer to the second question is probably negative.) In the background, a question previously raised in the literature (see **[10]**, **[6]**, **[7]**) is lurking: If D is a Prüfer domain, does D have the n-generator property for some positive integer n; more specifically, does D have the 2-generator property?

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