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COVERABLE SEMIGROUPS

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ABSTRACT. The concept of a semilattice having small semilattices has been studied and some equivalences of this property have been investigated. In the process of investigating semilattices, the author found a class of semigroups called coverable semigroups, and the interesting fact about it is that a necessary and sufficient condition for a compact semilattice to have small semilattices is to be coverable. Also by virtue of the concept of coverable semigroups, one can show that the usual interval is costable instead of the more intricate method of proving it by producing an arc.

1. Let S be a topological semigroup and U_0 be an open cover of S. We write $V_0 < U_0$ if V_0 is a cover of S such that V_0 refines U_0 and if $V, U \in V_0$, then there exists $W \in V_0$ such that $VU \subseteq W$. We shall say that S is (finitely) coverable if given an open cover U_0 of S, there exists a (finite) open cover V_0 such that $V_0 < U_0$. Furthermore, S is said to be (finitely) neighborable if given an open cover of S, there exists a (finite) collection V_0 of subsets of S such that $V_0 < U_0$ and $S = \bigcup_{V \in V_0} V^0$ where V^0 is the interior of V (i.e. union of all open sets contained in V). A compact semigroup S is costable if given any continuous homomorphism from a compact semigroup K onto S, then cd $K \geqq$ cd S where cd is the codimension with respect to some nondegenerate abelian group.

It is obvious that every finitely coverable semigroup is compact and neighborable. Any Hausdorff space given the left trivial multiplication is coverable. The following proposition is in relation to neighborable semigroups.

LEMMA 1.1. *If S is a neighborable semigroup, then S has small semigroups at $p = p^2$ (i.e. if p is in S and V is an open set containing p, then there exists a neighborhood N of p such that $N^2 \subseteq N \subseteq V$).*

PROOF. Let $p \in S$ and U be an open set containing p. Then $U_0 = \{U, S \backslash \{p\}\}$ is an open cover of S; hence there exists a cover V_0 of S such that $V_0 < U_0$ and $S = \bigcup_{V \in V_0} V^0$.

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Let $p \in V^0$ where $V \in V_0$. Then $VV \subseteq W_1$ for some $W_1 \in V_0$ and $V^3 \subseteq VW_1 \subseteq W_2$ for some $W_2 \in V_0$. By induction, one can show that if n is a positive integer, then $V^n \subseteq W$ for some $W \in V_0$. Since $V_0 < U_0$, then $W \subseteq U$ or $W \subseteq S \backslash \{p\}$, but $p \in V^n \subseteq W$ implies $W \subseteq U$. Hence $V^n \subseteq W \subseteq U$. But n is arbitrary; one can conclude $p \in V \subseteq \bigcup_{n \geqq 1} V^n \subseteq U$ and $\bigcup_{n \geqq 1} V^n$ is a subsemigroup.

In fact, for a compact semilattice the converse of 1.1 is true. Before we prove it, we need some other propositions. The following two lemmas are stated without proof.

LEMMA 1.2. *If S is a (finitely) coverable (or neighborable) semigroup and A is a closed subsemigroup, then A is (finitely) coverable (or neighborable).*

LEMMA 1.3 (A TOPOLOGICAL LEMMA). *If U_0 is an open cover of $\mathbf{P}_{i \in I} S_i$, where each S_i is a compact space, then there exists a finite subset J of I and for each i in J, there exists a finite open cover $\mathscr{V}_i$ of S_i such that $\{\mathbf{P}_{i \in J} V_i \times \mathbf{P}_{i \in I \backslash J} S_i \mid V_i \in \mathscr{V}_i\}$ refines U_0.*

LEMMA 1.4. *If $\{S_i\}_{i \in I}$ is a collection of compact semigroups, then $\mathbf{P}_{i \in I} S_i$ is (finitely) coverable if and only if each S_i is (finitely) coverable.*

PROOF. Suppose $\mathbf{P}_{i \in I} S_i$ is (finitely) coverable and $j \in I$. Let W_0 be an open cover of S_j. Then $U_0 = \{W \times \mathbf{P}_{i \neq j} S_i \mid W \in W_0\}$ is an open cover of $\mathbf{P}_{i \in I} S_i$. There exists V_0 a (finite) open cover of $\mathbf{P}_{i \in I} S_i$ such that $V_0 < U_0$. Then $\{\Pi_j(V) \mid V \in V_0\}$ is a (finite) open cover of S_j, where Π_j is the jth projection map. It is easy to check $\{\Pi_j(V) \mid V \in V_0\} < W_0$.

Suppose each S_i is (finitely) coverable and U_0 is an open cover of $\mathbf{P}_{i \in I} S_i$. By Lemma 1.3, there exists a finite subset J of I and open cover $\mathscr{V}_i$ where $i \in J$ such that $\{\mathbf{P}_{i \in J} V_i \times \mathbf{P}_{i \in I \backslash J} S_i \mid V_i \in \mathscr{V}_i\}$ refines U_0. But for each $i \in J$, there exists a (finite) open cover $\mathscr{O}_i$ such that $\mathscr{O}_i < \mathscr{V}_i$. Hence

$$\left\{\mathbf{P}_{i \in J} O_i \times \mathbf{P}_{i \in I \backslash J} S_i \mid O_i \in \mathscr{O}_i\right\} < \left\{\mathbf{P}_{i \in J} V_i \times \mathbf{P}_{i \in I \backslash J} S_i \mid V_i \in \mathscr{V}_i\right\}$$

would yield the result that $\{\mathbf{P}_{i \in J} O_i \times \mathbf{P}_{i \in I \backslash J} S_i \mid O_i \in \mathscr{O}_i\} < U_0$. Hence the product is (finitely) coverable.

LEMMA 1.5. *The min interval (i.e. $[0, 1]$ under the multiplication $xy = \min\{x, y\}$) is finitely coverable.*

PROOF. Let M be the min interval and U_0 be an open cover of M. There exists a Lebesgue number δ for U_0. Then M can be covered by a finite number of open intervals or half-open intervals, each of which has length δ. Furthermore, the product of any two such intervals is contained in the interval with the smaller left-hand endpoint. Hence M is finitely coverable.

THEOREM 1.6. *If S is a compact semilattice, then S is coverable if and only if it has small semilattices.*

PROOF. If S is coverable, by Lemma 1.1, S has small semilattices. Suppose S has small semilattices; then [6] showed that S is embeddable topologically and algebraically into a product of min intervals. Since min intervals are coverable, by Lemma 1.4 their product is coverable. By the compactness of S, S can be considered as a closed subsemilattice of the product of min intervals. Hence S is coverable by Lemma 1.2.

2. In this section, we will depart from semilattices and investigate coverable/neighborable semigroups in general. An interesting corollary is that the usual interval is costable.

LEMMA 2.1. *If S is a neighborable semigroup, then $H(e)$ is totally disconnected for each $e=e^2$.*

PROOF. Let $e=e^2 \in S$. Suppose $H(e)$ is not totally disconnected. Then there exists a compact connected nondegenerate subgroup G of $H(e)$. Since G is also neighborable, then G has small semigroups at e. Let N be a neighborhood of e in G such that N is a proper subsemigroup of G. Let U be a set open in G such that $U=U^{-1}$ and $e \in U \subseteq N$. Then $\bigcup_{n\geqq 1} U^n \subseteq N$. But this is a contradiction.

LEMMA 2.2. *Every compact totally disconnected semigroup is coverable.*

PROOF. Since every such semigroup is an inverse limit of finite semigroups [4] and finite semigroups are coverable, then an application of Lemmas 1.2 and 1.4 yields the conclusion.

The following theorem shows how neighborable semigroups behave under homomorphisms.

THEOREM 2.3. *If $f: S \to T$ is a continuous closed homomorphism from a (finitely) neighborable semigroup onto a fully normal semigroup T, then T is (finitely) neighborable.*

PROOF. Let C_0 be an open cover of T. Then there exists an open cover U_0 of T such that U_0 is a star-refinement of C_0. Since $\{f^{-1}(U) \mid U \in U_0\}$ is an open cover of S, there exists a (finite) cover V_0 of S such that $V_0 < \{f^{-1}(U) \mid U \in U_0\}$, and $S=\bigcup_{V\in V_0} V^0$.

For each $y \in T$, define $W_y=\{V \in V_0 \mid V \cap f^{-1}(y) \neq \varnothing\}$. Then $\{f(W_y) \mid y \in T\}$ is a (finite) cover of T since V_0 is a (finite) cover of S, and it is obvious that $y \in f(W_y)^0$ since f is closed. Now one has to show $\{f(W_y) \mid y \in T\} < C_0$.

Let $y \in T$. Fix $V_1 \in V_0$ such that $V_1 \cap f^{-1}(y) \neq \varnothing$. Then there exists $U_1 \in U_0$ such that $V_1 \subseteq f^{-1}(U_1)$, and $U_1^* \subseteq C$ for some $C \in C_0$ since U_0 is a star-refinement of C_0. Let $z \in f(W_y)$. Then there exists $V_2 \in V_0$ such that

$V_2 \cap f^{-1}(y) \neq \varnothing$ and $z \in f(V_2)$. But $V_2 \subseteq f^{-1}(U_2)$ for some $U_2 \in U_0$. Hence $y \in f(V_1) \cap f(V_2) \subseteq U_1 \cap U_2$. Since $U_1 \cap U_2 \neq \varnothing$, then $U_2 \subseteq U_1^* \subseteq C$. Thus $z \in C$ implies $f(W_y) \subseteq C$. Furthermore $f(W_y)f(W_z) \subseteq f(W_{yz})$. The proof is complete.

COROLLARY 2.4. *If $f: S \to T$ is a continuous closed homomorphism from a (finitely) neighborable fully normal semigroup S onto a semigroup T, then T is (finitely) neighborable.*

With a little computation, one can determine that the usual interval [0, 1] under real number multiplication is not neighborable. Hence we have the following corollary.

COROLLARY 2.5. *If $f: S \to I$ is a continuous homomorphism from a compact semigroup S onto the usual interval I, then S is not totally disconnected (i.e. I is costable.)*

PROOF. If S is totally disconnected, then S is coverable. By the previous lemma, I has to be neighborable but it is not. Hence S is not totally disconnected, i.e. codimension of $S \geqq 1$.

The interesting fact about coverable semigroups is that they can have a lot of idempotents (e.g. some semilattices) or they can have very few idempotents (e.g. $[0, \frac{1}{2}]$ under the usual real number multiplication). The author conjectures that a compact semigroup with zero as its only idempotent is coverable.

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