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THE H^p CLASSES FOR α-CONVEX FUNCTIONS

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ABSTRACT. Given $\alpha>0$, we determine the H^p class to which an α-convex function belongs.

Introduction. In this paper we continue the study ([4], [5], [6], [7]) of $\mathcal{M}_\alpha$, the class of α-convex functions. Our purpose is to obtain the H^p class to which a given α-convex function belongs.

DEFINITION. Let $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ be analytic in the unit disc D, with $(f(z)/z)f'(z)\neq 0$ there, and let α be a real number. Then $f(z)$ is said to be α-convex in D if and only if

$$\operatorname{Re}\left[(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right]>0. \tag{1}$$

It is known [5] that if $f(z)$ is α-convex then $f(z)$ is starlike and univalent in D. Moreover, if $\alpha\geqq 1$ then $f(z)$ is convex in D. In our investigation we shall employ the H^p results concerning starlike and convex functions [3]. We shall also make use of the following subclasses, first introduced by Reade [11], of the class of close-to-convex functions.

DEFINITION. Let K_β denote the class of functions $f(z)$, analytic in D, for which there exists a convex function $h(z)$ such that

$$|\arg(f'(z)/h'(z))|\leqq \beta\pi/2. \tag{2}$$

The following theorem can be found in [7].

THEOREM 1. *If $f(z)\in\mathcal{M}_\alpha$ $(0\leqq\alpha\leqq 1)$, then $f(z)\in K_{1-\alpha}$.*

PROOF. Since $f(z)\in\mathcal{M}_\alpha$ one easily checks that $f(z)(zf'(z)/f(z))^\alpha$ is starlike, so that

$$h(z)\equiv\int_0^z\frac{f(w)}{w}\left(\frac{wf'(w)}{f(w)}\right)^\alpha dw \tag{3}$$

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is convex. The result then follows from the identity

$$f'(z)/h'(z) = (zf'(z)/f(z))^{1-\alpha}$$

and the fact that $f(z)$ is starlike.

We now make the following two observations.

(a) If $f(z) \in \mathcal{M}_\alpha$ ($\alpha>0$), then $f(z)$ is continuous in $\bar{D}$ (if ∞ is allowed as a value) and assumes no finite value more than once.

(b) If $f(z) \in \mathcal{M}_\alpha$ ($\alpha>0$), and $L(r)$ denotes the length of $\{w=f(z):|z|=r\}$, then $L(r)=O(M(r))$ as $r\to 1$.

Pommerenke **[10]** obtained the results in (a) and (b) for $f(z) \in K_\beta$, $\beta<1$, and consequently by Theorem 1 they follow for $f(z) \in \mathcal{M}_\alpha$, $0<\alpha<1$. In case $\alpha \geqq 1$, $f(z)$ is convex and the results (a) and (b) are well known.

We further observe from (a) that none of the classes of starlike functions of a positive order (less than unity) is contained in $\mathcal{M}_\alpha$ for some positive α. This follows from the existence of bounded functions, starlike of any given order (less than unity), which do not extend continuously to $\bar{D}$ **[2]**. The question of the order of starlikeness for the class $\mathcal{M}_\alpha$ remains open.

THEOREM 2. *If $f(z) \in K_\beta$ and there exists a convex function $h(z)$, not of the form $h(z)=a+bz(1+ze^{i\tau})^{-1}$, such that $|\arg(f'(z)/h'(z))| \leqq \beta\pi/2$, then there exists $\varepsilon=\varepsilon(f)>0$ such that*

$$f(z) \in H^{1/(1+\beta)+\varepsilon} \quad \textit{and} \quad f'(z) \in H^{1/(2+\beta)+\varepsilon}.$$

PROOF. We first observe that if $\beta=0$ or $\beta=1$, these results are known **[3]**. Writing $f'(z)=h'(z)P(z)$ where $|\arg P(z)| \leqq \beta\pi/2$, it follows that $P(z) \in H^\lambda$, $\forall\lambda$, $\lambda<1/\beta$. Also, from Theorem 3 of **[3]**, $h'(z) \in H^{1/2+\delta}$ for some $\delta=\delta(h)>0$. Application of Hölder's inequality with

$$p = (\tfrac{1}{2} + \delta)(\beta + 2 - \delta),$$
$$q = (\tfrac{1}{2} + \delta)(\beta + 2 - \delta)(\beta/2 + \delta\beta + 3\delta/2 - \delta^2)^{-1}$$

yields

$$\int_{-\pi}^{\pi} |f'(z)|^{(\beta+2-\delta)^{-1}}\, d\theta \leqq \left(\int_{-\pi}^{\pi} |h'(z)|^{p/(\beta+2-\delta)}\, d\theta\right)^{1/p} \left(\int_{-\pi}^{\pi} |P(z)|^{q/(\beta+2-\delta)}\, d\theta\right)^{1/q}.$$

If δ is sufficiently small, each of the integrals on the right remains bounded as r tends to 1. Hence there exists $\varepsilon=\varepsilon(f)>0$ such that $f'(z) \in H^{1/(\beta+2)+\varepsilon}$. By a well-known theorem of Hardy-Littlewood **[1**, p. 88**]**, $f(z) \in H^{1/(\beta+1)+\varepsilon}$ for a possibly different value of ε.

We require the following integral representation **[6]** for functions in $\mathcal{M}_\alpha$, $\alpha>0$: the function $f(z)$ is in $\mathcal{M}_\alpha$, $\alpha>0$, if and only if there exists a starlike function $s(z)$ such that

$$f(z) = \left[\frac{1}{\alpha}\int_0^z [s(\zeta)]^{1/\alpha}\zeta^{-1}\,d\zeta\right]^\alpha. \tag{4}$$

If $\alpha>2$, $f(z)$ is bounded [**4**] and $f'(z)\in H^1$ (since $f(z)$ is convex).

Let us denote by $f_\alpha(z)$, the function obtained in (4) by letting $s(z)$ be the Koebe function, $k(z)=z(1-z)^{-2}$. It follows from (3) that $h(z)$ is of the form $z(1-ze^{i\tau})^{-1}$ if and only if $f(z)=e^{-i\tau}f_\alpha(ze^{i\tau})$. Theorems 1 and 2 then yield the following.

THEOREM 3. *If $f(z)\in\mathcal{M}_\alpha$ $(0\leqq\alpha\leqq 1)$ and is not a rotation of $f_\alpha(z)$, then there exists $\varepsilon=\varepsilon(f)>0$ such that*

$$f(z)\in H^{1/(2-\alpha)+\varepsilon} \quad \text{and} \quad f'(z)\in H^{1/(3-\alpha)+\varepsilon}.$$

We remark that for $0\leqq\alpha\leqq 2$, $f_\alpha(z)\notin H^{1/(2-\alpha)}$ (H^∞ if $\alpha=2$), although $f_\alpha(z)\in H^\lambda$, $\forall\lambda$, $\lambda<1/(2-\alpha)$.

We now wish to establish Theorem 3 for $1<\alpha\leqq 2$. In this case $f(z)$ is both convex and starlike. These geometric properties give rise to the usual Herglotz formulas, which in turn yield probability measures μ_1 and μ_2, respectively. We can suppose these measures to be normalized so that

$$\tfrac{1}{2}[\mu_i(t+0)+\mu_i(t-0)]=\mu_i(t), \quad \int_{-\pi}^{\pi}\mu_i(t)\,dt=0 \qquad (i=1,2).$$

The normalization determines μ_1 and μ_2 uniquely; we shall call them *the convex and starlike measures* associated with $f(z)$, respectively.

LEMMA 4. *Let $f(z)$ be an unbounded convex function and let μ_1 be the convex measure associated with $f(z)$. If the maximum jump of $\mu_1(t)$ is γ, then $f(z)\in H^\lambda$, $\forall\lambda$, $\lambda<1/(2\gamma-1)$, and $f'(z)\in H^\lambda$, $\forall\lambda$, $\lambda<1/2\gamma$.*

PROOF. We first observe that since $f(z)$ is unbounded, $\gamma\geqq\frac{1}{2}$ [**8**, pp. 67–72]. The proof of Lemma 4 then follows from [**3**, p. 345].

THEOREM 5. *Let $f(z)$ be an unbounded convex function. If the maximum jump of $\mu_1(t)$ is γ then the maximum jump in $\mu_2(t)$ is $\gamma-\frac{1}{2}$.*

PROOF. Since $f(z)$ is convex, either (a) $f(z)$ is continuous on $\bar{D}$ with the exception of one point, say z_0, and at this point, $f(z)\to\infty$ as $z\to z_0$ with $|z|\leqq 1$, $|z_0|=1$, or (b) $f(D)$ is a domain whose boundary is two parallel lines [**8**]. In case (b), the result of Theorem 5 is obvious; hence, we assume case (a). We also assume that the maximum jump, γ, in $\mu_1(t)$ occurs at $t=0$. It follows by the continuity remark in (a) that the maximum jump in $\mu_2(t)$, call it β, also occurs at $t=0$.

First, suppose $\gamma-\frac{1}{2}<\beta$. It is well known (see e.g. **[12]**) that $f(D)$ contains a sector of vertex angle $2\pi\beta$. Hence, for some constant C,

$$C((1+z)/(1-z))^{2\beta} \prec f(z), \qquad z \in D.$$

Thus, $\int_{-\pi}^{\pi} |f(z)|^{1/2\beta}\, d\theta$ becomes unbounded as r tends to 1, contradicting Lemma 4 (since $2\gamma-1<2\beta$).

We now suppose $\beta<\gamma-\frac{1}{2}$. Choosing δ such that $\beta<\delta<\gamma-\frac{1}{2}$, a result of Pommerenke **[9]** gives

$$M(r,f) = O((1-r)^{-2\delta}).$$

The Cauchy formula yields

$$M(r,f') = O((1-r)^{-2\delta-1}).$$

On the other hand, since $zf'(z)$ is starlike

$$|zf'(z)| \geqq K\,|z|/|1-z|^{2\gamma},$$

where K is a constant **[9]**. But this contradicts the fact that $2\delta+1<2\gamma$. The result follows.

THEOREM 6. *If $f(z) \in \mathcal{M}_\alpha$ $(0\leqq\alpha<2)$ and is not a rotation of $f_\alpha(z)$, then there exists $\varepsilon=\varepsilon(f)>0$ such that*

$$f(z) \in H^{1/(2-\alpha)+\varepsilon} \quad \text{and} \quad f'(z) \in H^{1/(3-\alpha)+\varepsilon}.$$

If $f(z) \in \mathcal{M}_2$ is unbounded, then $f(z)$ is a rotation of $f_2(z)$.

PROOF. By Theorem 3, we may assume $\alpha>1$. We also take $f(z)$ unbounded. Let μ_1 and μ_2 be as in Theorem 5. Since $f(z) \in \mathcal{M}_\alpha$, $\mu=(1-\alpha)\mu_2+\alpha\mu_1$ is a measure on $[-\pi, \pi]$. If γ denotes the maximum jump in μ_1, then the maximum jump in μ_2 occurs at the same point and equals $\gamma-\frac{1}{2}$. Thus, the maximum jump in μ is $\gamma+\frac{1}{2}(\alpha-1)$. The rotations of $f_\alpha(z)$ are given by equating this number with unity; i.e., by setting $\gamma=\frac{1}{2}(3-\alpha)$.

Thus, let us assume $\frac{1}{2}\leqq\gamma<\frac{1}{2}(3-\alpha)$. We observe that if $\alpha=2$ then $\gamma=\frac{1}{2}$, and consequently there is only one unbounded function in $\mathcal{M}_2$ (up to rotations).[2]

By Lemma 4, $f'(z) \in H^\lambda$ $(\lambda<1/2\gamma)$. Since $1/2\gamma>1/(3-\alpha)$ we may choose $\varepsilon=\varepsilon(f)>0$ so that $1/2\gamma>1/(3-\alpha)+\varepsilon$. Hence, $f'(z) \in H^{1/(3-\alpha)+\varepsilon}$, and, for perhaps a different ε, $f(z) \in H^{1/(2-\alpha)+\varepsilon}$.

[2] M. O. Reade has pointed out to the authors that this result is also obtained by considering $g(z)=\sqrt{(f(z^2))}$, which is odd and convex, whenever $f(z) \in \mathcal{M}_2$.

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