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ON MATROIDS ON EDGE SETS OF GRAPHS WITH CONNECTED SUBGRAPHS AS CIRCUITS

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ABSTRACT. It is proved that if $\mathscr{F}$ is a finite family of connected, finite graphs, then a graph G exists such that the subgraphs of G isomorphic to a member of the family cannot be regarded as the circuits of a matroid on the edge set of G.

1. In a recent paper [1] we have proved that there are only two matroids on the edge set of any graph G (let us call them edge set matroids), whose circuits are connected subgraphs which form homeomorphic equivalent classes. These matroids are the polygon-matroid, whose circuits are the cycles, and the matroid of bi-circular subgraphs, where a bi-circular graph is a graph formed by two cycles which either have a path in common, or a vertex in common, or are disjoint but linked by a path; these graphs are homeomorphic to those pictured in Figure 1.

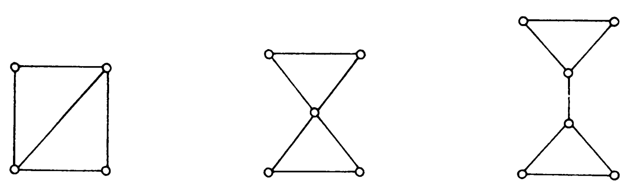

FIGURE 1

The hypothesis concerning homeomorphism is essential to the arguments in [1]. If we drop this hypothesis, the problem of finding all edge set matroids seems to be a very difficult one. As an unknown referee pointed out to me, a matroid of this kind is the matroid whose circuits are: (i) all cycles of even length; (ii) all graphs consisting of two cycles

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of odd length, having only one vertex in common; (iii) all graphs consisting of two cycles of odd length, joined by a path. In any graph G, the subgraphs of these kinds are the circuits of a matroid on the edge set of G but a cycle of odd length, although homeomorphic to a cycle of even length, is not a circuit of the matroid.

In this note we prove a theorem concerning edge set matroids. Our terminology is now slightly different from that used in **[1]**: we reserve the word "circuit" for the matroid-circuits and use "cycle" for simple closed paths in a graph. Moreover a matroid is defined as follows (see Whitney **[2]**):

Let E be a set of elements and $\mathscr{K}$ a family of subsets of E (circuits). $\mathscr{K}$ defines a matroid on E if and only if the following axioms hold:

AXIOM 1. No circuit is properly contained in another circuit.

AXIOM 2. If K and K' are distinct circuits, $a \in K \cap K'$ and $b \in K'-K$, then a circuit K'' exists such that $b \in K'' \subset K \cup K' - \{a\}$.

2. Let $\mathscr{F}$ be a family of abstract connected graphs such that in any graph G the subgraphs isomorphic to members of $\mathscr{F}$ are the circuits of a matroid. Call the members of $\mathscr{F}$ circuits. Then

LEMMA 1. *No circuit has a pendant edge.*

PROOF. Let K be a circuit with a pendant edge, say x. Take another circuit K', equal to K, and let $K \cup K'$ be such that x is the only edge common to K and K' and the pendant vertex of x in each circuit coincides with the vertex of higher degree in the other circuit. Clearly, x is a bridge in $K \cup K'$. By Axiom 2, $K \cup K' - \{x\}$ contains a circuit. But since all circuits must be connected, the existence of such a circuit contradicts Axiom 1. Thus the lemma is proved.

THEOREM 1. *Let $\mathscr{F}$ be a finite family of connected, finite graphs. Then a graph G exists such that the subgraphs of G isomorphic to a member of $\mathscr{F}$ (or, for brevity's, sake belonging to $\mathscr{F}$) cannot be regarded as the circuits of a matroid on the edge set of G.*

PROOF. Let $\mathscr{F}$ be a finite family of finite, connected graphs. The members of this family may eventually be regarded as the circuits of a matroid on the edge set of some graphs. However a graph G always exists with a subgraph which, according to the definition of a matroid, must also be a circuit but which does not belong to the family. This is a consequence from the fact that, for the members of a family $\mathscr{F}$ of connected, finite graphs to be circuits of an edge-set matroid defined on any graph G, there must always exist a member of $\mathscr{F}$ with a pair of edges of minimal distance arbitrarily large.

To prove it let K be a circuit, $\alpha=(a_1, a_2)$, $\beta=(b_1, b_2)$ two edges of K. Consider the four distances $d(a_i, b_j)$ for $i, j=1, 2$. Let r be the minimal distance between α and β, and suppose we choose a pair α, β in K for which this distance is maximal among all edge pairs. Moreover, without loss of generality, we may suppose $d(a_1, b_1)=r$. There are 6 distinct cases which are summarized in Table I (columns 1 to 5).

Cases	$d(a_1, b_1)$	$d(a_1, b_2)$	$d(a_2, b_1)$	$d(a_2, b_2)$	$d(a_1, b_1)$	$d(a_1, b_2)$	$d(a_2, b_1)$	$d(a_2, b_2)$
I	$r\ (>0)$	r	r	r	r	$r+1$	r	$r+1$
II	$r\ (>0)$	$r+1$	r	r	r	$r+1$	r	$r+1$
III	$r\ (>0)$	$r+1$	r	$r+1$	$r+1$	$r+1$	$r+1$	$r+1$
IV	$r\ (>0)$	$r+1$	$r+1$	r	r	$r+1$	$r+1$	$r+1$
V	$r\ (\geqq 0)$	$r+1$	$r+1$	$r+1$	r	$r+1$	$r+1$	$r+2$
VI	$r\ (\geqq 0)$	$r+1$	$r+1$	$r+2$	$r+1$	$r+1$	$r+2$	$r+2$

TABLE I

Take another circuit K' and let $K \cup K'$ be such that β is the only edge common to K and K'. For simplicity let us say the edges of K are black and those of K' are blue. Now, by Axiom 2, a circuit K'' exists such that $\alpha \in K'' \subset K \cup K' - \{\beta\}$. By Axiom 1, K'' contains both black and blue edges. Since K'' must be connected and as a consequence of Lemma 1, either K'' contains at least one blue path $P(b_1, b_2)$ with length $s \geqq 2$, or at least one of the vertices b_1 and b_2 is a cut-point of K'' and there exists at least one cycle in the blue block of K'' relative to this cut-point.

If blue paths exist, then take one with minimum length $s \geqq 2$. We distinguish two possibilities:

(a) $s \geqq 3$. Let β' be an edge of $P(b_1, b_2)$ incident to neither b_1 nor b_2. The minimal distance between α and β', which both belong to K'', is $\geqq r+1$, that is to say, we obtain a new circuit K'' from a given circuit K with a pair of edges α and β' whose minimal distance is greater than the minimal distance between the edges α and β of K.

(b) $s=2$. Let (b_1, x), (x, b_2) be the edges in $P(b_1, b_2)$. We have to examine the 6 cases of Table I. In cases I, II, IV and V, we set $\beta'=(b_1, x)$ and x plays now the role of b_2. In cases III and VI, we set $\beta'=(x, b_2)$ and x plays the role of b_1. The new distances between the endpoints of α and β' are given in the columns 6 to 9 of Table I. With this operation we obtain, in cases III and VI, a pair of edges in K'', namely α and β', whose minimal distance is larger than the distance between α and β. In the remaining cases, to obtain a circuit with a pair of edges satisfying this condition, one or two iterations of this operation may be required, each time with K'' and β' in the roles of K and β, respectively. In fact, cases I and II yield case III, case V yields case VI and case IV yields in a first iteration case V which in turn yields case VI. Now from cases III and VI, a new iteration allows us to achieve our aim.

If no blue path exists, then take the above mentioned blue cycle. Suppose the cycle belongs to the blue block of b_1. (The same argument holds a fortiori with b_2 instead of b_1.) Let β' be an edge of the cycle non-incident to b_1. Obviously, the minimal distance between α and β' is $\geqq r+1$.

Hence it is always possible to obtain from a pair of edges α, β in a circuit K, whose distance is r, a new pair α, β' in a circuit K'', whose distance is $\geqq r+1$. By repeating the argument, the theorem is proved.

Theorem 1 may also be stated more briefly as follows.

THEOREM 1′. *No edge-set matroid (on an arbitrary graph) may exist with a finite number of connected, finite graphs as circuits.*

REFERENCES

1. J. M. S. Simões-Pereira, *Subgraphs as matroid cells*, Math. Z. **127** (1972), 315–322.
2. H. Whitney, *On the abstract properties of linear dependence*, Amer. J. Math. **57** (1935), 501–533.

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