PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 38, Number 3, May 1973

THE COMMUTATOR SUBGROUP MADE ABELIAN

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ABSTRACT. A theorem on covering spaces is proved which yields the following information about a group π, its commutator subgroup π' and their abelianizations: If $\pi^{ab} \cong Z_{p^n}$, a cyclic group of order a power of the prime p, then $\pi'^{ab} = p\pi'^{ab}$. Hence if π is also finitely generated, then π'^{ab} is finite of order prime to p.

The purpose of this note is to prove the following theorem and some related results:

THEOREM 1. *Let X be a connected* CW *complex with $H_1(X) \cong Z_{p^n}$. Let X' be the normal p^n-fold covering space of X with transformation group Z_{p^n}. Then $H_1(X')$ is p-divisible. In particular, if $\pi_1 X$ is finitely generated, then $\pi_1 X'$ is also so $H_1(X')$ is finite of order prime to p.*

Notation. If π is a group, $\pi' = [\pi, \pi]$ is the commutator subgroup and $\pi^{ab} = \pi/\pi'$ is its abelianization.

An immediate corollary is

THEOREM 2. *If $\pi^{ab} \cong Z_{p^n}$, a cyclic group of order a power of the prime p, then π'^{ab} is p-divisible; i.e. $\pi'^{ab} = p\pi'^{ab}$.*

Theorem 2 follows from Theorem 1 by observing that the Eilenberg-Mac Lane space $K(\pi', 1)$ is the p^n-fold covering space of $K(\pi, 1)$ and $H_1(K(\pi, 1)) = \pi^{ab}$, $H_1(K(\pi', 1)) = \pi'^{ab}$.

The proof of Theorem 1 is based on the homology Serre Spectral Sequence of the fibration $X' \to X \to K(Z_{p^n}, 1)$: $E^2_{**} = H_*(Z_{p^n}; H_*(X'))$ (local coefficients based on the action of Z_{p^n} on X') converging to $H_*(X)$ (simple Z-coefficients).

Because it is a first quadrant spectral sequence there is an exact sequence

$$E^2_{2,0} \to E^2_{0,1} \to H_1(X) \to E^2_{1,0} \to 0.$$

Received by the editors March 14, 1972.

AMS (MOS) subject classifications (1970). Primary 20F35, 55A10; Secondary 18H10, 55B25.

Key words and phrases. Abelianization, commutator subgroup, covering space, homology of groups.

[1] This work was partially supported by a grant from the National Science Foundation.

But $E^2_{s,0}=H_s(Z_{p^n}; H_0(X'))=H_s(Z_{p^n})$ which is Z_{p^n} for $s=1$ and 0 for $s=2$. Since $H_1(X)\to H_1(Z_{p^n})$ is an isomorphism, we conclude that

$$H_0(Z_{p^n}; H_1(X')) = E^2_{0,1} = 0.$$

The theorem is proved once we show:

PROPOSITION. *If G is a finite p-group and $H_0(G; M)=0$ for some G-module M, then M is p-divisible.*

PROOF. $H_0(G; M)=M/IM$ where $I=\ker \varepsilon\colon Z[G]\to Z$ is the augmentation. So $H_0(G; M)=0$ means $M=IM$. The proposition will be proved if we can show that for some integer N, $I^N\subset pI$, whence $M=I^NM\subset pIM=pM$. This is equivalent to showing that $J=I\otimes Z_p=\ker \varepsilon\otimes Z_p$ is nilpotent. This is well known [**1**, p. 703] but for completeness we prove it here for the case $G=Z_{p^n}$: Let t be the generator of Z_{p^n} (written multiplicatively). $t^{p^n}=1$. Then $J=(t-1)Z_p[G]$. $J^{p^n}=(t-1)^{p^n}Z_p[G]$. But modulo p, $(t-1)^{p^n}\equiv(t^{p^n}-1)=0$ so $J^{p^n}=0$.

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