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THE ADDITIVE GROUP OF COMMUTATIVE RINGS GENERATED BY IDEMPOTENTS[1]

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ABSTRACT. If R is a ring, let R^+ denote its additive group. Our purpose is to give an elementary proof that if R is a commutative ring generated by idempotents, then any subring of R generated by idempotents is pure. This yields immediately an independent proof of the following result of G. M. Bergman. If R is a commutative ring with identity and if R is generated by idempotents, then R^+ is a direct sum of cyclic groups.

G. M. Bergman [**1**, Corollary 4.3] has proved (in particular) that if R is a commutative ring with identity generated by idempotents, then R^+ is a direct sum of cyclic groups. This result is rather striking inasmuch as it contains, as a special case when R^+ is torsion free, the recent celebrated result of G. Nöbeling [**2**] that the finite-valued functions from any set I to the integers Z form a free abelian group with respect to pointwise addition. Recall that the latter result was first proved for a countable set I by E. Specker [**3**] with the aid of the continuum hypothesis.

If S is a subset of the ring R, we shall use $\{S\}$ to denote the *subring* generated by S, while $\langle S\rangle$ denotes the *subgroup* of R^+ that S generates. An elementary but important fact concerning a commutative ring R generated by idempotents is the following. *If A is a finitely generated subring of R, then A^+ is a finitely generated subgroup of R^+.* Since a subgroup of a finitely generated commutative group is again finitely generated and since each element in R is a linear combination over Z of idempotent elements, it suffices to prove the above statement for the case that A is generated as a ring by a finite number of idempotents. However, in this case, A^+ is generated by all possible products of the idempotents generation A. Hence A^+ is finitely generated.

THEOREM 1. *Let R be a commutative ring generated by idempotents. Any subring of R generated by idempotents is pure.*

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THEOREM 2 (BERGMAN). *If R is a commutative ring generated by idempotents, then R^+ is a direct sum of cyclic groups.*

PROOF OF THEOREM 1. Let A be a subring generated by idempotents. The purity of A in R means that, for each prime p and positive integer n, the equation $p^n x=a$ where $a \in A$ has a solution in A whenever the equation has a solution in R. Thus it obviously suffices to prove the purity of A in case the subring A is generated (as a ring) by a *finite* number of idempotents $e_1, e_2, \cdots, e_n$. The purity of A is accomplished by induction on n. Equivalently, for an idempotent e, we shall prove that $\{A, e\}$ is pure under the assumption that A is a subring generated by n idempotents and that A is pure for any such subring. We can take $A=0$ to prove inductively the purity of a subring generated by a single nonzero idempotent.

For simplicity of notation, let $B=\{A, Ae\}$ and let $C=\{B, e\}=\{A, e\}$. Since $A(Ae)\subseteq Ae$ and since $Be\subseteq B$, we observe that $B=\langle A, Ae\rangle$ and $C=\langle B, e\rangle$. In order to prove that C is pure, we shall first prove that B is pure. For a prime p and a positive integer n, suppose that $p^n x=b$ where $b \in B$ and $x \in R$. Since $B=\langle A, Ae\rangle$, we can write $b=a_1+a_2e$ where $a_1, a_2 \in A$. Since $p^n(xe)=be=(a_1+a_2)e$ is contained in Ae and since Ae is a subring of R generated by the n idempotents $e_1e, e_2e, \cdots, e_ne$, the induction hypothesis asserts that $p^n(a_3e)=be$ for some $a_3 \in A$. Note that $p^n(x-a_3e)=a_1-a_1e$. Let $D=\{e_i-e_ie\}$ be the subring of R generated by the idempotents e_i-e_ie where $1\leqq i\leqq n$. Since $A=\{e_1, e_2, \cdots, e_n\}$ and since $a_1 \in A$, the verification that a_1-a_1e is contained in D is trivial if we make the observation that $\prod e_i-(\prod e_i)e=\prod (e_i-e_ie)$, where $\prod$ represents a product over any nonempty subset of $[1, 2, \cdots, n]$. By the induction hypothesis, we conclude that $p^n d=p^n(x-a_3e)$ for some $d \in D$ since D is generated by n idempotents and since $p^n(x-a_3e)=a_1-a_1e$ is in D. Thus $p^n x=p^n(a_3e+d)$, and we have proved the purity of B in R since $a_3e+d \in B$.

Based on the purity of B, we can now prove the purity of C. Suppose that $p^n x=c$ where $c \in C$ and $x \in R$. We want to show that $p^n c_1=c$ for some $c_1 \in C$. Write $c=b+p^m qe$ where $b \in B$ and $(q, p)=1$. Since $(q, p)=1$, q is not relevant to the equation under consideration and therefore there is no loss of generality in assuming that $q=1$, so now let $c=b+p^m e$. If $m\geqq n$, then $p^n b_1=b$ for some $b_1 \in B$ due to the purity of B. Thus $c_1=b_1+p^{m-n}e$ is the desired solution, $p^n c_1=c$. Therefore, we may assume that $m<n$. Hence let $r=n-m>0$. From the purity of B, we can choose $b_1 \in B$ so that $p^m b_1=b$. Suppose, for a positive integer i, that we have already shown (as we have for $i=1$) the existence of $x_i \in \{x\}$ and $b_i \in B$ such that

$$p^{ri+m}x_i = p^m b_i + p^m e. \tag{1}$$

Since e is idempotent, we obtain from (1) the equation

$$p^{ri}x_i(p^m e) = p^m b_i e + p^m e \tag{2}$$

and finally

$$p^{r(i+1)+m}x_{i+1} = p^m b_{i+1} + p^m e, \tag{3}$$

where $x_{i+1}=p^{r(i-1)}x_i^2$ and $b_{i+1}=b_i^2+2b_ie$. Therefore, (1) has a solution, for all $i \geqq 1$, with $x_i \in \{x\}$ and $b_i \in B$. This implies that $p^m e+B^+$ has infinite p-height in $\{C, x\}^+/B^+$. From a remark in the introduction, we know that $\{C, x\}^+$ is a finitely generated group. Hence $\{C, x\}^+/B^+$ is a direct sum of cyclic groups since it is finitely generated. We conclude that $p^m e+B^+$ has finite order t where $(t, p)=1$ since a direct sum of cyclic groups has no elements of infinite p-height except elements of finite order relatively prime to p. Since $(t, p)=1$, we may replace the equation $p^n x= b+p^m e$ by the equation $p^n y=tb+tp^m e$. However, the latter equation has a solution in B since $tp^m e \in B$ and B is pure. We have shown that C is pure, and this completes the proof of Theorem 1.

The proof of Theorem 2 now follows quickly by induction on the (minimal) cardinality of a generating set of idempotents. Let E be a set of idempotents that generate the ring R, and assume that E has been chosen so that $|E|$ is as small as possible. If E is finite, then R^+ is finitely generated and, therefore, is a direct sum of cyclic groups. Thus assume that $|E|=\aleph \geqq \aleph_0$. In view of the purity of any subring of R generated by idempotents and the fact that direct sums of cyclic groups are the pure-projectives in the category of abelian groups, it suffices to prove that $\{A, e\}^+/A^+$ is a direct sum of cyclic groups whenever e is an idempotent and A is a subring of R generated by idempotents. The point here is that we can ascend to R with a chain

$$0 = A_0 \subset A_1 \subset \cdots \subset A_\alpha \subset \cdots$$

of subrings such that: (1) $A_{\alpha+1}$ is a simple extension of A_α by an idempotent, (2) $A_\beta=\bigcup_{\alpha<\beta} A_\alpha$ if β is a limit, and (3) A_α is generated by fewer than $\aleph$ idempotents. To prove that $\{A, e\}^+/A^+$ is a direct sum of cyclic groups, we again insert $B=\{A, Ae\}$. Since $B=\langle A, Ae\rangle$, we have the isomorphism $B^+/A^+ \cong (Ae/A \cap Ae)^+$ because $A \cap Ae$ is an ideal of Ae. Moreover, $Ae/A \cap Ae$ is a commutative ring generated by fewer than $\aleph$ idempotents, for by hypothesis the same is true of A. By the induction hypothesis, $(Ae/A \cap Ae)^+$ is a direct sum of cyclic groups. Since B is pure in $C= \{A, e\}=\langle B, e\rangle$ and since $C^+/B^+ \cong \langle e\rangle^+/(B \cap \langle e\rangle)^+$ is cyclic, B^+ is a direct summand of C^+. Therefore, C^+/A^+ is a direct sum of cyclic groups since B^+/A^+ is. This completes the proof of Theorem 2.

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