PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 38, Number 3, May 1973

SUFFICIENT CONDITIONS FOR PERIODICITY OF A KILLING VECTOR FIELD

WALTER C. LYNGE

ABSTRACT. Let X be a complete Killing vector field on an n-dimensional connected Riemannian manifold. Our main purpose is to show that if X has as few as n closed orbits which are located properly with respect to each other, then X must have periodic flow. Together with a known result, this implies that periodicity of the flow characterizes those complete vector fields having all orbits closed which can be Killing with respect to some Riemannian metric on a connected manifold M. We give a generalization of this characterization which applies to arbitrary complete vector fields on M.

THEOREM. *Let X be a complete Killing vector field on a connected, n-dimensional Riemannian manifold M. Assume there are n distinct points p, $p_1, \cdots, p_{n-1}$ in M such that the respective orbits γ, $\gamma_1, \cdots, \gamma_{n-1}$ of X through them are closed and γ is nontrivial. Suppose further that each p_i is joined to p by a unique minimizing geodesic and $d(p, p_i)=\eta_i<D/2$, where d denotes distance on M and D is the diameter of γ as a subset of M. Let $w_1, \cdots, w_{n-1}$ be the unit vectors in T_pM such that* $\exp \eta_i w_i = p_i$, $i=1, \cdots, n-1$. *Assume that the vectors X_p, $w_1, \cdots, w_{n-1}$ span T_pM. Then the flow φ_t of X is periodic.*

PROOF. Fix i in $\{1, 2, \cdots, n-1\}$. We first show that the orbit γ_i does not lie entirely in the sphere $S_p(\eta_i)$ of radius η_i about p. For suppose it did. Let q be a point in γ such that $d(p, q)=D$. Let s be a real number such that $\varphi_s(p)=q$. Then $d(p, \varphi_s(p_i))=\eta_i$ by our supposition. Since φ_s is an isometry, $d(q, \varphi_s(p_i))=\eta_i$. By the triangle inequality, $d(p, q)\leqq d(p, \varphi_s(p_i))+d(\varphi_s(p_i), q)$. Thus $D\leqq 2\eta_i$. This contradicts the hypothesis of the theorem.

Let λ and λ_i denote the prime periods of γ and γ_i respectively, assuming for the moment that γ_i is nontrivial. We claim that λ_i/λ is rational. Suppose it is not. Let s be any number in the interval $(0, \lambda_i)$. We recall that it is possible to find integers m and n such that $|m\lambda-n\lambda_i-s|$ is as small as we please. Furthermore, since γ_i is closed, the map from the reals to M given by $t\to\varphi_t(p_i)$ is uniformly continuous. Let $\varepsilon>0$ be given. Then

Received by the editors June 28, 1972.

AMS (MOS) subject classifications (1970). Primary 53C20, 53B20; Secondary 57D25.

Key words and phrases. Killing vector field, isometry, flow, Lie algebra, Riemannian metric, Riemannian manifold, closed orbit, minimizing geodesic.

there exists $\delta>0$ such that $d(\varphi_{m\lambda}(p_i), \varphi_{n\lambda_i+t}(p_i))<\varepsilon$ whenever

$$|m\lambda - (n\lambda_i + t)| < \delta.$$

Now given any s in $(0, \lambda_i)$, pick m and n so that $|m\lambda-(n\lambda_i+s)|<\delta$. Then $d(\varphi_{m\lambda}(p_i), \varphi_{n\lambda_i+s}(p_i))<\varepsilon$. But $\varphi_{n\lambda_i+s}(p_i)=\varphi_s(p_i)$, since $n\lambda_i$ is a multiple of the period of γ_i. Thus $d(\varphi_{m\lambda}(p_i), \varphi_s(p_i))<\varepsilon$. Moreover, $\varphi_{m\lambda}(p_i)$ lies in $S_p(\eta_i)$, since $d(p, \varphi_{m\lambda}(p_i))=d(\varphi_{m\lambda}(p), \varphi_{m\lambda}(p_i))=d(p, p_i)=\eta_i$. We have shown that given any point $\varphi_s(p_i)$ in γ_i and given any $\varepsilon>0$, there exists a point in $S_p(\eta_i)$, namely $\varphi_{m\lambda}(p_i)$, such that $d(\varphi_{m\lambda}(p_i), \varphi_s(p_i))<\varepsilon$. Thus $\gamma_i\subset S_p(\eta_i)$, contradicting what we proved in the previous paragraph. Hence λ_i/λ is rational.

We write $n_i\lambda_i=m_i\lambda$ for integers n_i, m_i. We have $p_i=\varphi_{n_i\lambda_i}(p_i)=\varphi_{m_i\lambda}(p_i)$. Let N be the least common multiple of the m_i as i ranges through the indices for which γ_i is nontrivial. If all the γ_i are trivial, take N to be 1. Then we have $\varphi_{\lambda N}(p_i)=p_i$, $i=1, \cdots, n-1$. Since also $\varphi_{\lambda N}(p)=p$, we have that every point of the unique minimizing geodesic from p to p_i is left fixed by $\varphi_{\lambda N}$. Thus $\varphi_{\lambda N*}(p)w_i=w_i$, $i=1, \cdots, n-1$. Since also $\varphi_{\lambda N*}(p)X_p=X_p$, we have that $\varphi_{\lambda N*}(p)$ is the identity on T_pM. Hence $\varphi_{\lambda N}$ is the identity on M [**1**, p. 62]. Hence the flow φ_t of X is periodic.

A special case of this theorem is: Assuming X is nontrivial, the flow of X is periodic if there is an open set U such that all the orbits of X which meet U are closed. We remark that this special case can be proved from a different and more extrinsic point of view than the theorem itself. Let H be the closure in the isometry group G of M of the 1-parameter subgroup of G corresponding to X. Then $H\approx T^m$ for some $m\geqq 1$. By considering the action of H on M, it can be shown that $m=1$. We omit the details.

It is known that if X is a complete vector field having periodic flow on a connected manifold M, then there is a Riemannian metric g on M such that X is Killing with respect to g. From this and the theorem, it follows that periodicity of the flow characterizes those complete vector fields having all orbits closed which can be Killing with respect to some Riemannian metric on M. In conclusion, we note that this characterization can easily be generalized to apply to arbitrary complete vector fields on M as follows:

PROPOSITION. *Let X be a complete vector field on a connected manifold M. Then there exists a Riemannian metric g on M such that X is Killing with respect to g if and only if there exist (for some positive integer m) complete vector fields $X_1, \cdots, X_m$ on M such that*

(i) *The flows of the X_i are periodic;*

(ii) $[X_i, X_j]=0$, $i, j=1, \cdots, m$;

(iii) $X=\sum_{i=1}^m c_iX_i$, *c_i real numbers.*

PROOF. Suppose X is Killing with respect to a metric g. Let G be the isometry group of (M, g). We recall that the Lie algebra $\mathfrak{g}$ of G is naturally isomorphic with the Lie algebra of complete Killing vector fields on M. Given Y in $\mathfrak{g}$, we denote by Y^* the corresponding complete Killing vector field. Now let A in $\mathfrak{g}$ be such that $A^*=X$. If $\exp tA$ is a closed 1-parameter subgroup of G, then the flow of X is periodic and we are finished. If $\exp tA$ is not closed, let H be its closure in G. Then H is isomorphic to some toral group T^m. Let $\{A_1, \cdots, A_m\}$ be a basis for the Lie algebra of H such that $\exp tA_i$, $i=1, \cdots, m$, are closed 1-parameter subgroups of H. Then $A=\sum_{i=1}^m d_i A_i$ for real numbers d_i. Set $X_i=A_i^*$ and $c_i=d_i$, $i=1, \cdots, m$. (i), (ii), and (iii) are obviously satisfied.

To prove the sufficiency, let φ^i be the flow of X_i, and let λ_i denote the period of φ^i. Let $\tilde{g}$ be any Riemannian metric on M and define a new metric g by

$$g(u, v) = \int_0^{\lambda_m}\int_0^{\lambda_{m-1}} \cdots \int_0^{\lambda_1} \tilde{g}_{\varphi_t(p)}(\varphi_{t*}(p)u, \varphi_{t*}(p)v)\, dt^1\, dt^2 \cdots dt^m$$

where p is a point of M, u, v are vectors in T_pM, and $\varphi_t=\varphi^1_{t^1} \circ \cdots \circ \varphi^m_{t^m}$. Since $[X_i, X_j]=0$, $i, j=1, \cdots, m$, the flows φ^i commute with each other. From this it follows that each φ^i is an isometric flow with respect to g, and hence each X_i is Killing with respect to g. Thus X is also Killing with respect to g.

BIBLIOGRAPHY

1. Sigurdur Helgason, *Differential geometry and symmetric spaces*, Pure and Appl. Math., vol. 12, Academic Press, New York, 1962. MR **26** #2986.

DEPARTMENT OF MATHEMATICS, EDGECLIFF COLLEGE, CINCINNATI, OHIO 45206

Current address (Walter C. Lynge): 6501 N. Brookwood Lane, Peoria, Illinois 61614