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THE PRODUCT OF COMMUTING CONICAL PROJECTIONS IS A PROJECTION

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ABSTRACT. The statement in the title is proved for two projections on closed convex cones in Hilbert space.

The projection on a closed convex set K in a real Hilbert space H is the operator P_K assigning to each $x \in H$ the nearest point in K. In a recent paper devoted to the study of these operators **[1]** the author formulated the conjecture that the product of any two commuting projections is a projection again, namely the projection on the intersection of the corresponding convex sets [**1**, p. 325], and proved its validity for projections on finite dimensional closed convex cones [**1**, Theorem 5.6]. In the present article the original proof is extended to any two commuting projections on cones, eliminating the finite dimensionality requirement. In addition included is a brief proof, already implicit in our previous paper [**1**, Lemma 5.3], that commutativity is also necessary for the products of two such projections to be projections.

The discussion, entirely elementary, is based on the following simple facts: The variational inequalities for the problem of minimizing the distance to a closed convex set K—known as the Bourbaki-Cheney-Goldstein inequalities—are

$$(1) \qquad \langle x - P_K x, P_K x - y \rangle \geqq 0, \qquad \forall\, y \in K,$$

and characterize $P_K x$ entirely, that is,

$$(2) \quad \{z = P_K x\} \Leftrightarrow \{\langle x - z, z - y \rangle \geqq 0, z \in K, \forall\, y \in K\} \quad [\mathbf{1}, \text{Lemma 1.2}].$$

Setting $y = P_K x'$ in (1) and adding the resulting inequality to that obtained by interchanging x and x', one deduces

$$(3) \qquad \langle x - x', P_K x - P_K x' \rangle \geqq \|P_K x - P_K x'\|^2, \qquad \forall\, x, x' \in H,$$

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whence it follows by the Schwarz inequality

$$\text{(4)} \qquad \|P_K x - P_K x'\| \leqq \|x - x'\|, \qquad \forall\, x, x' \in H.$$

Thus, projections are nonexpansive mappings. Another simple consequence of (2) to be used in the sequel is

$$\text{(5)} \qquad P_K(P_K x + t(x - P_K x)) = P_K x, \qquad \forall\, t \geqq 0, \forall\, x \in H.$$

Should K be a closed convex cone C with vertex at 0 then (1) yields the equation

$$\text{(6)} \qquad \langle x - P_C x, P_C x\rangle = 0, \qquad \forall\, x \in H,$$

obtained by replacing y successively by 0 and $2P_C x$.

Thus prepared we may now proceed to the statement and proof of our theorem:

THEOREM. *The products, in either order, of two projections in Hilbert space on closed convex cones with vertices at the origin are both projections if and only if the given projections commute. In such a case the common value of the products is the projection on the intersection of the cones.*

PROOF. Necessity and the last part of the theorem result at once from the proposition

$$\text{(7)} \qquad \{P_{C_1} P_{C_2} = P_{C_3}\} \Rightarrow \{C_3 = C_1 \cap C_2\},$$

whose proof runs as follows: Clearly $C_3 \supset C_1 \cap C_2$. On the other hand if $x \in C_3$ then by (6),

$$\begin{aligned} \|x - P_{C_2} x\|^2 &= \langle x - P_{C_2} x, x\rangle \\ &= -\langle P_{C_2} x - P_{C_1} P_{C_2} x, P_{C_1} P_{C_2} x\rangle = 0, \end{aligned}$$

that is, $x \in C_2$. Hence $C_3 \subset C_2$, and since $C_3 \subset C_1$, $C_3 \subset C_1 \cap C_2$. Thus $C_1 \cap C_2 = C_3$. To prove sufficiency it must be shown that if

$$\text{(8)} \qquad P_{C_1} P_{C_2} = P_{C_2} P_{C_1}$$

then $P_{C_1} P_{C_2} = P_{C_1 \cap C_2}$. Thus we assume (8) and write for any $x \in H$, $x_1 = P_{C_1} x$, $u_1 = x - P_{C_1} x$. We then consider the points

$$x(s, t) = t(x_1 + su_1) + (1 - t)P_{C_2}(x_1 + su_1), \qquad 0 \leqq s, t \leqq 1.$$

One checks first, with the assistance of (5) and (8), that

$$\text{(9)} \qquad \begin{aligned} P_{C_2} P_{C_1} x(s, t) &= P_{C_1} P_{C_2} x(s, t) = P_{C_1} P_{C_2}(x_1 + su_1) \\ &= P_{C_2} P_{C_1}(x_1 + su_1) = P_{C_2} x_1, \end{aligned}$$

and then, on use of (6), that

$$
\begin{aligned}
&\langle x(s,t) - P_{C_1}x(s,t), P_{C_1}P_{C_2}x(s,t)\rangle \\
&\quad = \langle x(s,t) - P_{C_1}P_{C_2}x(s,t), P_{C_1}P_{C_2}x(s,t)\rangle \\
&\qquad - \langle P_{C_1}x(s,t) - P_{C_2}P_{C_1}x(s,t), P_{C_2}P_{C_1}x(s,t)\rangle \\
&\quad = t\langle x_1 - P_{C_1}P_{C_2}x(s,t), P_{C_1}P_{C_2}x(s,t)\rangle \\
&\qquad + (1-t)\langle P_{C_2}(x_1+su_1) - P_{C_1}P_{C_2}x(s,t), P_{C_1}P_{C_2}x(s,t)\rangle \\
&\qquad + st\langle u_1, P_{C_2}x_1\rangle \\
&\quad = t\langle x_1 - P_{C_2}x_1, P_{C_2}x_1\rangle \\
&\qquad + (1-t)\langle P_{C_2}(x_1+su_1) - P_{C_1}P_{C_2}(x_1+su_1), P_{C_1}P_{C_2}(x_1+su_1)\rangle \\
&\qquad + st\langle u_1, P_{C_2}x_1\rangle \\
&\quad = st\langle u_1, P_{C_2}x_1\rangle.
\end{aligned} \tag{10}
$$

Moreover, since $P_{C_2}x_1 = P_{C_1}P_{C_2}x_1 \in C_1$,

$$x(0,t) = \lim_{s\downarrow 0} x(s,t) = tx_1 + (1-t)P_{C_2}x_1 \in C_1,$$

and since projections are nonexpansive mappings (cf. (4)),

$$
\begin{aligned}
&\|s^{-1}(x(s,t) - P_{C_1}x(s,t))\| \\
&\quad \leqq s^{-1}\|x(s,t) - x(0,t)\| = \|tu_1 + (1-t)s^{-1}(P_{C_2}(x_1+su_1) - P_{C_2}x_1)\| \\
&\quad \leqq t\|u_1\| + (1-t)s^{-1}\|P_{C_2}(x_1+su_1) - P_{C_2}x_1\| \\
&\quad \leqq t\|u_1\| + (1-t)\|u_1\| \leqq \|u_1\|.
\end{aligned}
$$

Hence, $s^{-1}(x(s,t)-P_{C_1}x(s,t))$ being bounded, it is possible to find, for any t in the closed interval $[0, 1]$, a positive sequence $s_n\downarrow 0$ such that $s_n^{-1}(x(s_n,t)-P_{C_1}x(s_n,t))$ converge weakly to a limit u_t. From (10) one obtains

$$\langle u_t, P_{C_2}x_1\rangle = t\langle u_1, P_{C_2}x_1\rangle, \qquad 0 \leqq t \leqq 1. \tag{11}$$

Moreover, by (1),

$$\langle s_n^{-1}(x(s_n,t) - P_{C_1}x(s_n,t)), P_{C_1}x(s_n,t) - y\rangle \geqq 0, \qquad \forall\, y \in C_1,$$

whence letting $n\to\infty$,

$$\langle u_t, x(0,t) - y\rangle \geqq 0, \qquad \forall\, y \in C_1. \tag{12}$$

In particular, replacing y by tx_1 and by $tx_1+2(1-t)P_{C_2}x_1$, both belonging to C_1,

$$(1-t)\langle u_t, P_{C_2}x_1\rangle \geqq 0, \qquad -(1-t)\langle u_t, P_{C_2}x_1\rangle \geqq 0. \tag{13}$$

Now, assuming, as one may, that $0<t<1$, one derives from (13) and (11), $\langle u_1, P_{C_2}x_1\rangle=0$, that is

$$\langle x - P_{C_1}x, P_{C_2}P_{C_1}x\rangle = 0, \qquad \forall\, x \in H. \tag{14}$$

But then, since P_{C_1} and P_{C_2} commute,

$$\begin{aligned}
&\langle x - P_{C_1}P_{C_2}x, P_{C_1}P_{C_2}x - y\rangle \\
&\quad = \langle x - P_{C_1}x, P_{C_1}P_{C_2}x - y\rangle + \langle P_{C_1}x - P_{C_1}P_{C_2}x, P_{C_1}P_{C_2}x - y\rangle \\
&\quad = \langle x - P_{C_1}x, -y\rangle + \langle P_{C_1}x - P_{C_2}P_{C_1}x, P_{C_2}P_{C_1}x - y\rangle \\
&\quad = -\langle x - P_{C_1}x, P_{C_1}x\rangle + \langle x - P_{C_1}x, P_{C_1}x - y\rangle \\
&\qquad + \langle P_{C_1}x - P_{C_2}P_{C_1}x, P_{C_2}P_{C_1}x - y\rangle,
\end{aligned}$$

and by (1) and (6),

$$\langle x - P_{C_1}P_{C_2}x, P_{C_1}P_{C_2}x - y\rangle \geqq 0, \qquad \forall\, y \in C_1 \cap C_2,$$

that is,

$$P_{C_1}P_{C_2}x = P_{C_1\cap C_2}x, \qquad \forall\, x \in H,$$

by (2). Q.E.D.

Reference

1. E. H. Zarantonello, *Projections on convex sets in Hilbert space and spectral theory*, Contributions to Nonlinear Functional Analysis, Academic Press, New York, 1971, pp. 237–424.

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