

## ON A NONLINEAR STOCHASTIC INTEGRAL EQUATION OF THE HAMMERSTEIN TYPE

W. J. PADGETT

**ABSTRACT.** A nonlinear stochastic integral equation of the Hammerstein type in the form

$$x(t; \omega) = h(t; \omega) + \int_S k(t, s; \omega) f(s, x(s; \omega)) d\mu(s)$$

is studied where  $t \in S$ , a  $\sigma$ -finite measure space with certain properties,  $\omega \in \Omega$ , the supporting set of a probability measure space  $(\Omega, \mathcal{A}, P)$ , and the integral is a Bochner integral. A random solution of the equation is defined to be a second order vector-valued stochastic process  $x(t; \omega)$  on  $S$  which satisfies the equation almost certainly. Using certain spaces of functions, which are spaces of second order vector-valued stochastic processes on  $S$ , and fixed point theory, several theorems are proved which give conditions such that a unique random solution exists.

**1. Introduction.** The purpose of this note is to study the existence and uniqueness of a *random solution* of a nonlinear stochastic integral equation of the Hammerstein type of the form

$$(1.1) \quad x(t; \omega) = h(t; \omega) + \int_S k(t, s; \omega) f(s, x(s; \omega)) d\mu(s),$$

where

(i)  $S$  is a locally compact metric space with metric  $d$  defined on  $S \times S$  and  $\mu$  is a complete  $\sigma$ -finite measure defined on the collection of Borel subsets of  $S$ ;

(ii)  $\omega \in \Omega$ , where  $\Omega$  is the supporting set of the probability measure space  $(\Omega, \mathcal{A}, P)$ ;

(iii)  $x(t; \omega)$  is the unknown vector-valued random variable for each  $t \in S$ ;

(iv)  $h(t; \omega)$  is the stochastic free term defined for  $t \in S$ ;

(v)  $k(t, s; \omega)$  is the stochastic kernel defined for  $t$  and  $s$  in  $S$ ; and

(vi)  $f(t, x)$  is a vector-valued function of  $t \in S$  and  $x$ .

The integral in equation (1.1) is interpreted as a Bochner integral [12].

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Further assumptions concerning the functions in (1.1) will be stated in §2.

The equation (1.1) is a generalization of stochastic integral equations studied by Padgett and Tsokos [9], Tsokos [11], and Anderson [1]. Also, equation (1.1) is a stochastic version of the deterministic integral equations which were investigated by Petryshyn and Fitzpatrick [10], Browder and Gupta [5], Browder, de Figueiredo, and Gupta [6], among others.

In order to investigate the stochastic integral equation (1.1), we will define several spaces of functions which are spaces of second order vector-valued stochastic processes on  $S$  and will use certain aspects of the "theory of admissibility" of Banach spaces as introduced into the study of integral equations by Corduneanu [7] and the methods of "probabilistic functional analysis" [3].

**2. Preliminaries.** We will further assume that  $S$  is the union of a countable family of compact subsets  $\{C_n\}$  having the properties that  $C_1 \subset C_2 \subset C_3 \subset \dots$  and that for any other compact set in  $S$  there is a  $C_i$  which contains it [2].

We define  $C = C(S, L_2(\Omega, A, P))$  to be the space of all continuous functions from  $S$  into the space  $L_2(\Omega, A, P)$  with the topology of uniform convergence on compacta. That is, for each fixed  $t \in S$ ,  $x(t; \omega)$  is a vector-valued random variable such that

$$\|x(t; \omega)\|_{L_2(\Omega, A, P)}^2 = \int_{\Omega} |x(t; \omega)|^2 dP(\omega) < \infty.$$

It may be noted that  $C(S, L_2(\Omega, A, P))$  is a locally convex space [12, pp. 24–26] whose topology is defined by the countable family of seminorms given by

$$\|x(t; \omega)\|_n = \sup_{t \in C_n} \|x(t; \omega)\|_{L_2(\Omega, A, P)}, \quad n = 1, 2, \dots$$

Moreover,  $C(S, L_2(\Omega, A, P))$  is complete relative to this topology since  $L_2(\Omega, A, P)$  is complete.

We further define  $BC = BC(S, L_2(\Omega, A, P))$  to be the Banach space of all bounded continuous functions from  $S$  into  $L_2(\Omega, A, P)$  with norm

$$\|x(t; \omega)\|_{BC} = \sup_{t \in S} \|x(t; \omega)\|_{L_2(\Omega, A, P)}.$$

The space  $BC \subset C$  is the space of all second order vector-valued stochastic processes defined on  $S$  which are bounded and continuous in mean-square.

We will consider the functions  $h(t; \omega)$  and  $f(t, x(t; \omega))$  to be in the space  $C(S, L_2(\Omega, A, P))$ . With respect to the stochastic kernel we assume that for each pair  $(t, s)$ ,  $k(t, s; \omega) \in L_{\infty}(\Omega, A, P)$  and denote the norm by

$$\|k(t, s; \omega)\| = \|k(t, s; \omega)\|_{L_{\infty}(\Omega, A, P)} = P\text{-ess sup}_{\omega \in \Omega} |k(t, s; \omega)|.$$

Also, we will suppose that  $k(t, s; \omega)$  is such that

$$\| \|k(t, s; \omega)\| \| \cdot \|x(s; \omega)\|_{L_2(\Omega, A, P)}$$

is  $\mu$ -integrable with respect to  $s$  for each  $t \in S$  and  $x(s; \omega)$  in  $C(S, L_2(\Omega, A, P))$ , and that there exists a real-valued function  $G$  defined  $\mu$ -a.e. on  $S$  so that  $G(s) \|x(s; \omega)\|_{L_2(\Omega, A, P)}$  is  $\mu$ -integrable and, for each pair  $(t, s) \in S \times S$ ,

$$\| \|k(t, u; \omega) - k(s, u; \omega)\| \| \cdot \|x(u; \omega)\|_{L_2(\Omega, A, P)} \leq G(u) \|x(u; \omega)\|_{L_2(\Omega, A, P)}$$

$\mu$ -a.e. Further, for almost all  $s \in S$ ,  $k(t, s; \omega)$  will be continuous in  $t$  from  $S$  into  $L_\infty(\Omega, A, P)$ .

We now define the integral operator  $T$  on  $C(S, L_2(\Omega, A, P))$  by

$$(2.1) \quad (Tx)(t; \omega) = \int_S k(t, s; \omega)x(s; \omega) d\mu(s),$$

where the integral is a Bochner integral. From the conditions on  $k(t, s; \omega)$ , we have that for each  $t \in S$ ,  $(Tx)(t; \omega) \in L_2(\Omega, A, P)$  and that  $(Tx)(t; \omega)$  is continuous in mean square by Lebesgue's dominated convergence theorem. That is,  $(Tx)(t; \omega) \in C(S, L_2(\Omega, A, P))$ .

LEMMA 2.1. *The linear operator  $T$  defined by equation (2.1) is continuous from  $C(S, L_2(\Omega, A, P))$  into itself.*

PROOF. Note that  $C(S, L_2(\Omega, A, P))$  is a Fréchet space with metric  $d^*$  defined by the Fréchet combination of the sequence of seminorms  $\|\cdot\|_n$ ,  $n=1, 2, \dots$ .

Define the sequence of linear operators  $\{T_M\}$ ,  $M=1, 2, \dots$ , by

$$(T_M x)(t; \omega) = \int_{C_M} k(t, s; \omega)x(s; \omega) d\mu(s).$$

Hence, as  $M \rightarrow \infty$  we have  $(T_M x)(t; \omega) \rightarrow (Tx)(t; \omega)$ .

Let  $\{x_j(t; \omega)\}$  be a sequence of functions converging to  $x(t; \omega)$  in  $C(S, L_2(\Omega, A, P))$ . Then by definition of the seminorms, for each  $M$

$$\begin{aligned} & \| (T_M x)(t; \omega) - (T_M x_j)(t; \omega) \|_n \\ & \leq \sup_{t \in C_n} \int_{C_M} \| \|k(t, s; \omega)\| \| \cdot \|x(s; \omega) - x_j(s; \omega)\|_{L_2(\Omega, A, P)} d\mu(s). \end{aligned}$$

Since  $\|x(s; \omega) - x_j(s; \omega)\|_{L_2(\Omega, A, P)} \rightarrow 0$  uniformly on the compact set  $C_M$ , for  $\varepsilon > 0$  there exists a positive integer  $N_M$  such that  $j \geq N_M$  implies

$$\| (T_M x)(t; \omega) - (T_M x_j)(t; \omega) \|_n < \varepsilon \sup_{t \in C_n} \int_{C_M} \| \|k(t, s; \omega)\| \| d\mu(s).$$

Now, by the conditions on  $k(t, s; \omega)$ , there exists a constant  $K_n$  such that  $\|k(t, s; \omega)\| \leq K_n$  for all  $t \in C_n$  and almost all  $s$ . Hence, for  $j \geq N_M$

$$\|(T_M x)(t; \omega) - (T_M x_j)(t; \omega)\|_n < \varepsilon K_n \mu(C_M).$$

Since convergence in every seminorm is equivalent to convergence in the metric  $d^*$ ,  $(T_M x_j)(t; \omega)$  converges to  $(T_M x)(t; \omega)$  in  $C(S, L_2(\Omega, A, P))$  for each  $M$ . Therefore, by [8, p. 54],  $T$  is continuous from  $C(S, L_2(\Omega, A, P))$  into itself.

Let  $B$  and  $D$  be Banach spaces. The pair  $(B, D)$  is said to be *admissible* with respect to a linear operator  $T$  if  $T(B) \subset D$ .

**LEMMA 2.2.** *If  $T$  is a continuous linear operator from  $C(S, L_2(\Omega, A, P))$  into itself and  $B, D \subset C(S, L_2(\Omega, A, P))$  are Banach spaces stronger than  $C(S, L_2(\Omega, A, P))$  such that  $(B, D)$  is admissible with respect to  $T$ , then  $T$  is continuous from  $B$  into  $D$ .*

The lemma follows from the closed-graph theorem.

From Lemmas 2.1 and 2.2 it follows that  $T$  defined by equation (2.1) is a bounded linear operator from  $B$  into  $D$ .

By a *random solution* of the equation (1.1) we will mean a function  $x(t; \omega)$  in  $C(S, L_2(\Omega, A, P))$  which satisfies the equation  $P$ -a.e.

**3. Existence of a random solution.** We now present theorems concerning the existence and uniqueness of a random solution of the equation (1.1).

**THEOREM 3.1.** *We consider the stochastic integral equation (1.1) subject to the following conditions:*

(i)  *$B$  and  $D$  are Banach spaces stronger than  $C(S, L_2(\Omega, A, P))$  such that  $(B, D)$  is admissible with respect to the integral operator defined by equation (2.1);*

(ii)  *$x(t; \omega) \rightarrow f(t, x(t; \omega))$  is an operator from the set*

$$Q(\rho) = \{x(t; \omega) : x(t; \omega) \in D, \|x(t; \omega)\|_D \leq \rho\}$$

*into the space  $B$  satisfying the Lipschitz condition*

$$\|f(t, x(t; \omega)) - f(t, y(t; \omega))\|_B \leq \lambda \|x(t; \omega) - y(t; \omega)\|_D$$

*for  $x(t; \omega), y(t; \omega) \in Q(\rho)$ , where  $\rho$  and  $\lambda$  are constants;*

(iii)  *$h(t; \omega) \in D$ .*

*Then there exists a unique random solution of (1.1) in  $Q(\rho)$ , provided  $\lambda K < 1$  and*

$$\|h(t; \omega)\|_D + K \|f(t, 0)\|_B \leq \rho(1 - \lambda K),$$

*where  $K$  is the norm of  $T$ .*

PROOF. Define the operator  $U$  from  $Q(\rho)$  into  $D$  by

$$(Ux)(t; \omega) = h(t; \omega) + \int_S k(t, s; \omega) f(s, x(s; \omega)) d\mu(s).$$

Then from the conditions of the theorem

$$\begin{aligned} \|(Ux)(t; \omega)\|_D &\leq \|h(t; \omega)\|_D + K \|f(t, x(t; \omega))\|_B \\ &\leq \|h(t; \omega)\|_D + K \|f(t, 0)\|_B + K\lambda \|x(t; \omega)\|_D \leq \rho. \end{aligned}$$

Hence,  $(Ux)(t; \omega) \in Q(\rho)$ .

Now, for  $x(t; \omega), y(t; \omega) \in Q(\rho)$  we have by condition (ii) that

$$\begin{aligned} \|(Ux)(t; \omega) - (Uy)(t; \omega)\|_D &= \left\| \int_S k(t, s; \omega) [f(s, x(s; \omega)) - f(s, y(s; \omega))] d\mu(s) \right\|_D \\ &\leq K \|f(t, x(t; \omega)) - f(t, y(t; \omega))\|_B \\ &\leq \lambda K \|x(t; \omega) - y(t; \omega)\|_D. \end{aligned}$$

Since  $\lambda K < 1$ ,  $U$  is a contraction on  $Q(\rho)$ .

Therefore, by Banach's fixed point theorem there exists a unique  $x^*(t; \omega) \in Q(\rho)$  which is a fixed point of  $U$ , that is,  $x^*(t; \omega)$  is the unique random solution of equation (1.1).

A similar theorem may be obtained when  $f$  is a nonlinear contraction on  $Q(\rho)$  [4].

**THEOREM 3.2.** *Assume that the stochastic integral equation (1.1) satisfies the following conditions:*

- (i) *same as Theorem 3.1(i);*
- (ii)  *$x(t; \omega) \rightarrow f(t, x(t; \omega))$  is an operator from the set  $Q(\rho)$  into the space  $B$  satisfying*

$$\|f(t, x(t; \omega)) - f(t, y(t; \omega))\|_B \leq \phi(\|x(t; \omega) - y(t; \omega)\|_D)$$

*for  $x(t; \omega), y(t; \omega) \in Q(\rho)$ , where  $\phi$  is a real-valued continuous function such that  $\phi(s) < s$  for  $s > 0$ ;*

- (iii)  *$h(t; \omega) \in D$ .*

*Then there exists a unique random solution of (1.1) in  $Q(\rho)$ , provided  $K \leq 1$  and  $\|h(t; \omega)\|_D + K \|f(t, 0)\|_B \leq \rho(1 - K)$ , where  $K$  is the norm of  $T$ .*

The proof of Theorem 3.2 is similar to that of Theorem 3.1 except that the fixed point theorem of Boyd and Wong [4] is used.

The following is a useful application of Theorem 3.1.

COROLLARY 3.1. *Suppose the stochastic integral equation (1.1) satisfies the following conditions:*

- (i)  $\sup_{t \in S} \int_S \|k(t, s; \omega)\| d\mu(s) < \infty$ ;
- (ii)  $f(t, x)$  is a continuous function of  $t \in S$  uniformly in  $x$  such that for  $\|x(t; \omega)\|_{BC}, \|y(t; \omega)\|_{BC} \leq \rho$

$$\|f(t, x(t; \omega)) - f(t, y(t; \omega))\|_{L_2(\Omega, A, P)} \leq \lambda \|x(t; \omega) - y(t; \omega)\|_{L_2(\Omega, A, P)}$$

for each  $t \in S$ , where  $\lambda$  and  $\rho$  are constants;

- (iii)  $h(t; \omega)$  is a bounded continuous function from  $S$  into  $L_2(\Omega, A, P)$ .

Then there exists a unique random solution of equation (1.1), provided  $\sup_{t \in S} \int_S \|k(t, s; \omega)\| d\mu(s)$ ,  $\lambda$ , and  $\|f(t, 0)\|_{BC}$  are sufficiently small.

PROOF. We must show that condition (i) implies that the pair  $(BC, BC)$  is admissible with respect to the integral operator  $T$  defined by equation (2.1). Let  $x(t; \omega) \in BC(S, L_2(\Omega, A, P))$ . Then by the properties of the Bochner integral

$$\begin{aligned} \|(Tx)(t; \omega)\|_{L_2(\Omega, A, P)} &\leq \int_S \|k(t, s; \omega)x(s; \omega)\|_{L_2(\Omega, A, P)} d\mu(s) \\ &\leq \sup_{t \in S} \|x(t; \omega)\|_{L_2(\Omega, A, P)} \int_S \|k(t, s; \omega)\| d\mu(s) \\ &\leq \|x(t; \omega)\|_{BC} \sup_{t \in S} \int_S \|k(t, s; \omega)\| d\mu(s). \end{aligned}$$

Hence,  $(Tx)(t; \omega) \in BC(S, L_2(\Omega, A, P))$ , that is,  $(BC, BC)$  is admissible with respect to  $T$ .

Conditions (ii)–(iii) clearly imply that conditions (ii)–(iii) of Theorem 3.1 hold. Thus, by Theorem 3.1, there exists a unique random solution of equation (1.1).

Other corollaries of Theorems 3.1 and 3.2 may be obtained by choosing different Banach spaces contained in the space  $C(S, L_2(\Omega, A, P))$  and different conditions on  $f$  and  $k$ .

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH CAROLINA, COLUMBIA, SOUTH CAROLINA 29208