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A CLASS OF WILD CLOSED CURVES THAT SPAN ORIENTABLE SURFACES[1]

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ABSTRACT. A classical result in the topology of manifolds asserts that every polygonal closed curve in three-space bounds an orientable surface. In this note we relax the condition that the curve be locally tame and obtain a partial generalization.

THEOREM 1. *If a simple closed curve k is a boundary component of an annulus A that is untwisted and unknotted and if $k=\bigcap_{i=1}^{\infty} T_i$, where T_i is a solid torus, then k bounds a disk.*

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PRELIMINARIES. Since we are not in the PL category it is necessary to explain what is meant by saying A is untwisted and unknotted. By the Bing Approximation Theorem [**2**], we may take A locally polyhedral mod k and each $\dot{T}_i$ ($=$boundary of T_i) polyhedral. If k' is a polygonal nonbounding curve on $A\backslash k$ near k, then k' and the other boundary component l of A determines a polyhedral annulus to which the criteria of Kyle applies [**6**]. In essence it is assumed these annuli are untwisted and unknotted.

STATEMENT I. Hypothesis as in Theorem 1; conclusion: k bounds an orientable surface. (The conclusion that k bounds an orientable surface holds even if A is knotted.)

PROOF OF STATEMENT I. The simple closed curves k and l are the boundary components of A. Since $k=\bigcap_1^\infty T_i$, there is an integer N such that $\dot{T}_N$ separates k and l. By general position considerations we assume

$$A \cap \dot{T}_N = (A\backslash k) \cap \dot{T}_N = s_1 \cup \cdots \cup s_m,$$

is a nonempty collection of pairwise disjoint simple closed curves. By a familiar disk replacement argument, no s_i that is ~ 0 on both $\dot{T}_N$ and A need occur. If $s_i \nsim 0$ on A but $s_i \sim 0$ on $\dot{T}_N$ we easily find the desired disk

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by replacing a circular ring on A (not containing k) by a disk of $\dot{T}_N$. The problem is thus reduced to the case no $s_i \sim 0$ on $\dot{T}_N$. By the fact that $s_1, \cdots, s_m$ are disjoint, all s_i's are longitudes or nonlongitudes.

Consider the case all s_i's are nonlongitudes. Then each s_i links k. (Although k is not a polyhedral core of T_N, there is a polyhedral core as close to k as we like.) Now $s_i \sim l$ on $A \backslash k$. Hence l links k and hence l links any sufficiently good approximation to k. But the annulus determined by l and such an approximation to k is untwisted by hypothesis. This is impossible, and each s_i must be a longitude. By definition of longitude, there is a surface B such that $s_i = \partial B$, $B \subset S^3 \backslash T_N$. If s_i separates A into A' and A'', we may suppose A' contains k and $A' \cap (S^3 \backslash T_N) = \square$. Then $B \cup A'$ is the desired surface. This proves Statement I.

It remains to show that the surface may be taken to be a disk under the full hypothesis of Theorem 1 regarding A. The tori $T_1, T_2, \cdots$ by Theorem 2 below may be assumed concentric (see [3] or [5] for definition of concentric). Let us assume the tori are concentric for the moment and finish the proof that k bounds a disk.

We must show some T_N (say T_1) is unknotted. Since T_1 and T_2 are concentric, $\mathrm{Cl}(T_1 \backslash T_2)$ contains an annulus Z with one boundary component on $\dot{T}_1$ and one on $\dot{T}_2$. Since l and k are exterior to $\dot{T}_1$ and interior to $\dot{T}_2$, respectively, a Z can be found that is a subset of A. The fact that Z is unknotted and untwisted means its boundary components can approximate a polygonal core of T_1 arbitrarily closely and therefore T_1 must be unknotted. Thus the surface B can be taken to be a disk and k bounds a (possibly wild) disk. (See examples of Alford [1] and Gillman [4].)

THEOREM 2. *If k is a boundary component of an annulus A that is untwisted and unknotted and if $k = \bigcap_1^\infty T_i$, where T_i is a solid torus, then for some subsequence $i_1, i_2, \cdots$ of the integers $T_{i_{k+1}}$ is concentric to T_{i_k}.*

REMARKS. As mentioned above each T_i may be assumed polyhedral. If solid tori B and B^* are such that $B \subset \operatorname{int} B^*$, then B and B^* are concentric iff $\mathrm{Cl}(B^* \backslash B) = S^1 \times S^1 \times [0, 1]$. An integer $O(B^*, B)$ is defined by Schubert [8] for any such pair of solid tori. Since $k = \bigcap T_i$, and A is untwisted and unknotted we are concerned only with the cases $O(T_i, T_j) = \pm 1$, $i \neq j$. By a criterion of Edwards [3], concentricity of B and B^* in the presence of $O(B, B^*) = \pm 1$ is equivalent to the equivalence of the knot-types of the center lines of B and B^*.

PROOF OF THEOREM 2. Suppose, on the contrary, there is a subsequence $i_1, i_2, \cdots$ such that $T_{i_{k+1}}$ is knotted in T_{i_k}, $k = 1, 2, \cdots$. By passing to a subsequence, we may suppose $i_k = k$. Thus T_2 is knotted in T_1, T_3 in T_2, $\cdots$. Let T_k have a polygonal center line l_k. Each l_k has a unique decomposition (apart from the order) into prime nontrivial knots. There is a subannulus of A with boundary components l_k and l_{k+j}. Now the subannulus of A with these boundary components is untwisted and unknotted, by hypothesis. Therefore the total curvature of each l_k and l_{k+j} is 2π. But it is well known that a polygonal knotted curve, i.e. (l_k), has a total curvature $\geqq 4\pi$ [7].

References

1. W. R. Alford, *Some "nice" wild 2-spheres in E^3*, Topology of 3-Manifolds and Related Topics (Proc. The Univ. of Georgia Inst., 1961), Prentice-Hall, Englewood Cliffs, N.J., 1962, pp. 29–33. MR **25** #4504.

2. R. H. Bing, *Approximating surfaces with polyhedral ones*, Ann. of Math. (2) **65** (1957), 456–483. MR **19**, 300.

3. C. H. Edwards, Jr., *Concentric solid tori in the 3-sphere*, Trans. Amer. Math. Soc. **102** (1962), 1–17. MR **25** #3514.

4. D. S. Gillman, *Sequentially* 1-ULC *tori*, Trans. Amer. Math. Soc. **111** (1964), 449–456. MR **28** #5433.

5. O. G. Harrold, Jr., H. C. Griffith and E. E. Posey, *A characterization of tame curves in three-space*, Trans. Amer. Math. Soc. **79** (1955), 12–34. MR **19**, 972.

6. R. H. Kyle, *Embeddings of Möbius bands in 3-dimensional space*, Proc. Roy. Irish Acad. Sect. A **57** (1955), 131–136. MR **19**, 976.

7. J. W. Milnor, *On the total curvature of knots*, Ann. of Math. (2) **52** (1950), 248–257. MR **12**, 273.

8. H. Schubert, *Knoten und Vollringe*, Acta Math. **90** (1953), 131–286. MR **17**, 291.

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