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GROUPS WITH CENTRAL 2-SYLOW INTERSECTIONS OF RANK AT MOST ONE

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ABSTRACT. An involution in a finite group is called central if it lies in the center of a 2-Sylow subgroup of G. A 2-Sylow intersection is called central if it is either trivial or contains a central involution. Suppose G is a finite simple group all of whose central 2-Sylow intersections are trivial or rank one 2-groups. It is proved that G is a known simple group.

1. **Introduction.** All groups considered in this paper are finite. An involution is called *central* if it lies in the center of some 2-Sylow subgroup of G. An intersection of distinct 2-Sylow groups is called a *central* 2-*Sylow intersection* if it is trivial or contains a central involution.

The objective of this paper is to prove the following.

THEOREM. *Let G be a simple group. Suppose all central 2-Sylow intersections of G have 2-rank at most one. Then G is isomorphic to one of the following groups*:

(i) $PSL(2, q)$, $q=2^n>2$,
(ii) $Sz(q)$, $q=2^n\geqq 8$,
(iii) $PSU(3, q)$, $q=2^n>2$,
(iv) $PSL(2, q)$, $q\equiv 3$ *or* 5 (mod 8), $q>5$,
(v) J, *the Janko group of order* 175,560.

In this paper, the groups of type (i), (ii) and (iii) are called "simple Bender groups". Throughout, $S(G)$ denotes the largest solvable normal subgroup of G. The rest of the notation used is standard.

2. **Previous results.** Of basic importance is

THEOREM 2.1. *Let z be a central involution in G. Suppose z lies in no proper 2-Sylow intersection of G. Then $\langle z^G\rangle$, the normal closure of z in G, has a center of odd order and, modulo this center, is the direct product of simple Bender groups and a 2-nilpotent group having 2-Sylow subgroup of exponent 2. z projects nontrivially on each of the components of this factor.*

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This is basically the theorem in [**3**], where it is proved that z satisfies the hypotheses of the fusion theorem of [**8**] (see also [**2**, p. 62], for a statement of the fusion theorem).

THEOREM 2.2. *Let G be a simple group with all central 2-Sylow intersections having rank at most one. Suppose for some central involution z in G, $C_G(z)$ is solvable. Then G is isomorphic to either $SL(2, 2^a)$, $Sz(2^n)$ (n odd), $U(3, 2^a)$, or $PSL(2, q)$, $q \equiv 3$ or 5 (mod 8).*

This is the main theorem of [**4**].

THEOREM 2.3. *Let G be a simple group with all central 2-Sylow intersections cyclic. Then G is isomorphic to $SL(2, 2^a)$, $Sz(2^n)$ (n odd), $U(3, 2^a)$, $PSL(2, q)$, $q \equiv 3$ or 5 (mod 8) or J, Janko's group of order 175,560.*

This is the principal theorem of [**5**]. It generalizes a theorem of Mazurov [**6**] which characterizes simple groups with *all* proper 2-Sylow intersections cyclic.

REMARK. The proof of the theorem of the introduction appears in the next section, and utilizes all of the results just quoted. There exists a somewhat longer proof which proceeds from "first principles" in the sense that dependence on [**4**], [**5**] and Mazurov's theorem is avoided. This proof requires, however, a "weak closure" characterization of Janko's group [**9**].

It should be remarked that Mazurov's theorem has also been generalized in the direction of considering groups, *all* of whose proper 2-Sylow intersections have rank at most one. The classification of groups bearing this hypothesis was first accomplished by Michael Aschbacher [**0**], and independently by Peter Landrock [**7**]. In both of these papers G is not assumed to be simple.

3. **Proof of the theorem.** Let G be a counterexample, that is, G satisfies *Hypothesis* 3.1.

(a) *G is a simple group*;

(b) *all central 2-Sylow intersections of G have rank at most one*;

(c) *G is not isomorphic to one of the groups $SL(2, 2^a)$, $Sz(2^n)$ (n odd), $U(3, 2^a)$, $PSL(2, q)$, $q \equiv 3$ or 5 (mod 8) or J.*

The proof proceeds by a series of short steps showing that Hypothesis 3.1 is inconsistent.

(i) *The centralizers of all central involutions of G are nonsolvable.*

This follows at once from Theorem 2.2.

(ii) *G possesses a quaternion central 2-Sylow intersection.*

Otherwise, by Theorem 2.3, G must be a group forbidden by Hypothesis 3.1(c).

(iii) *Define the following sets*:

$\mathscr{M}=\{N_G(V)|V$ *is a 2-subgroup of G containing a central involution. $N_G(V)$ is nonsolvable and contains a 2-Sylow subgroup of G*$\}$.

$\mathscr{N}=\{H|H\in\mathscr{M},\ |O_2(H)|\geqq|O_2(H_1)|$ *for all* $H_1\in\mathscr{M}\}$, *the elements of* $\mathscr{M}$ *with* O_2 *as large as possible.*

$\mathscr{N}^*=\{$*the maximal elements of* $\mathscr{N}\}$.

$\mathscr{N}^*$ *is not empty.*

It suffices to show $\mathscr{M}$ is nonempty. But by (i), if z is any central involution, $C_G(z)\in\mathscr{M}$.

(iv) *Let H be a fixed element of* $\mathscr{N}^*$. *All proper 2-Sylow intersections of H are central 2-Sylow intersections of G and so are rank one.*

Since H lies in $\mathscr{N}^*$, $|H|_2=|G|_2$ and $H=N_G(V_0)$ where V_0 is a nontrivial 2-group containing a central involution of G. Thus all proper 2-Sylow intersections of H are central 2-Sylow intersections of G.

(v) *Set* $S=S(H)$, *the largest normal solvable subgroup of H. Let V be a 2-Sylow subgroup of S. Then* $H=N_G(V)$ *and* $V=O_2(H)$ *is a rank one 2-group.*

By a Frattini argument, $H=SN_H(V)$ and so $N_G(V)$ is nonsolvable and $|N_G(V)|_2=|N_H(V)|_2=|H|_2=|G|_2$. Thus $N_G(V)$ lies in $\mathscr{M}$. Now

$$O_2(N_G(V))\geqq V\geqq O_2(S(H))=O_2(H).$$

Since H lies in $\mathscr{N}$, $O_2(N_G(V))=V=O_2(H)$ and this forces $H=N_G(V)$, $V=O_2(H)$. Since H (being nonsolvable) is not 2-closed, V lies in a proper 2-Sylow intersection of H and this intersection is rank one by (iv). Hence V has rank one.

(vi) *Let z be the unique involution in V. Then for any element x of G,* $z^x=x^{-1}zx$ *lies in the center of every 2-Sylow subgroup of G which contains it.*

Some conjugate z^g *lies in* $H-\{z\}$. *In the group* $\bar{H}=H/S$, $\bar{z}^g$ *is a central involution which does not belong to any proper 2-Sylow intersection of* $\bar{H}$.

Suppose P is a 2-Sylow subgroup of G containing z in its center. Suppose z^x lies in $a^{-1}Pa$. Then if z^x were not in the center of $a^{-1}Pa$, $\langle z^x, z^a\rangle$ would be a fours-group lying in the intersection $a^{-1}Pa\cap(xc)^{-1}P(xc)$ where c is an appropriately chosen element of $C_G(z^x)$. Hypothesis 3.1(b) forces $a^{-1}Pa=c^{-1}x^{-1}Pxc$, so $a^{-1}xc\in N_G(a^{-1}Pa)$. But z^a and $z^x=z^{xc}$ are conjugate under the action of $a^{-1}xc$. Thus z^x must also lie in the center of $a^{-1}Pa$ contrary to assumption. Thus $z^x\in Z(a^{-1}Pa)$ and the first assertion is proved.

Now let P be a 2-Sylow subgroup of H. Then, since G is simple (Hypothesis 3.1(a)), Glauberman's Z^*-theorem **[1]** requires the existence of a conjugate z^g in $P-\{z\}$. Then since z is the unique involution in $P\cap S=V$ (by (v)), z^g does not lie in S. Since z^g lies in $Z(P)$ by the first part of (vi), we see that $\bar{z}^g$ is a central involution of $\bar{H}$. Now suppose $\bar{z}^g$ could be realized as an element in a proper 2-Sylow intersection $\bar{R}_1 \cap \bar{R}_2$ of $\bar{H}$. Then there exist 2-Sylow subgroups R_1 and R_2 of H, each containing the element z^g, such that $R_i S/S = \bar{R}_i$. Then since $R_i \cap S = V$, $R_1 \cap R_2$ contains the fours-group $\langle z^g, z\rangle$ contrary to Hypothesis 3.1(b). Thus $\bar{z}^g$ cannot lie in a proper 2-Sylow intersection of $\bar{H}$.

(vii) *Let $\bar{L}$ be the normal closure of $\bar{z}^g$ in $\bar{H}$. Then $\bar{L}$ is a direct product of Bender groups.*

Because of (vi) we may apply Theorem 2.1 to $\bar{H}$ and the involution $\bar{z}^g$. Since H/S has no nontrivial normal solvable subgroups, $S(\bar{L})=1$ and so $\bar{L}$ is semisimple.

(viii) *$\bar{L}$ is a simple Bender group.*

Since a 2-Sylow intersection in $\bar{H}$ can be realized as the homomorphic image of a 2-Sylow intersection of H, by (iv), such intersections in $\bar{H}$ have rank at most 2. Suppose $\bar{L}$ is not a simple Bender group. Then we must have $\bar{L} \simeq L_1 \times L_2$ where $L_i \simeq SL(2, 4)$ or $U(3, 4)$. Let $\bar{T}$ be a 2-Sylow center of $\bar{L}$ and choose T as 2-Sylow subgroup of the preimage of $\bar{T}$.

Let $\bar{T}_i$ denote the 2-Sylow center of L_i, $i=1, 2$, and let T_i be the preimage of $\bar{T}_i$ in the homomorphism $T \to TS/S = \bar{T}$. Then $T = T_1 T_2$, $T_1 \cap T_2 = T \cap S = V$. Each T_i lies in a proper 2-Sylow intersection of H, and has sectional 2-rank at least 2. Thus each T_i is generalized quaternion. Thus $Z(T) = Z(T_1 T_2) \leqq T \cap S = V$. But this is impossible since T is generated by V and conjugates of z^g which necessarily lie in $Z(T) - V$ by (vi).

(ix) *$\bar{H}/\bar{L}$ is 2-closed. $\bar{L}$ is the unique minimal normal subgroup of $\bar{H}$. $C_{\bar{H}}(\bar{L})=1$.*

Let T be a 2-Sylow subgroup of the preimage T of a 2-Sylow center of $\bar{L}$ in H, containing a conjugate z^g of z, outside of $T \cap S = V$. Then by (vi), $\langle z^g, z\rangle$ is a fours-group and so by Hypothesis 3.1(b), $N_G(T)$ is 2-closed. But if L is the preimage of $\bar{L}$ in H, $N_H(T)L = H$ so $\bar{H}/\bar{L}$ is a homomorphic image of $N_H(T)$ and is therefore 2-closed.

Let $\bar{K}$ be a minimal normal subgroup of $\bar{H}$ distinct from $\bar{L}$. Then $\bar{K} \cap \bar{L} = 1$ and so $\bar{K}$ is isomorphic to a subgroup of $\bar{H}/\bar{L}$ and is therefore 2-closed. But then $\bar{K} \leqq S(\bar{H}) = 1$, a contradiction.

Since $C_{\bar{H}}(\bar{L})$ is a normal subgroup of $\bar{H}$ meeting $\bar{L}$ trivially, it must also be trivial, since $\bar{L}$ is the unique minimal normal subgroup of $\bar{H}$.

(x) *$\bar{H}/\bar{L}$ has odd order.*

As before let L denote the preimage of $\bar{L}$ in H and let T be the 2-Sylow subgroup of the preimage of a 2-Sylow center of $\bar{L}$. Then as we have seen $N_G(T)$ is 2-closed. Let P be a 2-Sylow subgroup of H containing T. Then T is normal in P. The conjugates of z lying in $T-V$ span T/V and are central in P. Thus conjugation by elements in P induces a 2-group of automorphisms of $\bar{L}$ each of which:

(a) centralizes a 2-Sylow center $\bar{T}$ of $\bar{L}$, and

(b) centralizes $N_{\bar{L}}(\bar{T})/(\bar{P}\cap\bar{L})$, a cyclic group of order $q^{\delta}-1$, $q=|\bar{T}|$, $\delta=1$, 1, or 2.

Condition (b) occurs because $\bar{P}\leqq N_{\bar{H}}(\bar{T})$ and the latter is 2-closed (being a section covered by $N_G(T)$). All automorphisms of the Bender groups satisfying conditions (a) and (b) are inner. Thus we obtain the factorization

$$\bar{P} = C_{\bar{P}}(\bar{L}) \times (\bar{P} \cap \bar{L}).$$

By (ix) the first factor is trivial. Thus $\bar{P}\leqq\bar{L}$ and so (x) holds.

(xi) *V has order* 2.

V is a rank 1 2-group with unique involution z. Let P be a 2-Sylow subgroup of H. By (x), $PS/S=\bar{P}$ is a 2-Sylow subgroup of the Bender group $\bar{L}$. There exists a conjugate z^g in $P-V$. Since z and z^g both lie in the center of P we may assume without loss of generality that $g\in N_G(P)$. Then V^g is a normal subgroup of P with unique involution z^g. Thus $V^g\cap V=1$ and $V^gS/S=\bar{V}^g$ is a rank 1 normal subgroup of a 2-Sylow subgroup $\bar{P}$ of a Bender group. But the only such normal subgroups of a 2-Sylow subgroup of a Bender group have order 2. Hence $|V^g|=2$. This implies (xi).

(xii) *Let P be a 2-Sylow subgroup of H. Then by* (x), *$\bar{P}=PS/S$ is a 2-Sylow subgroup of the Bender group $\bar{L}$. Let $\bar{T}$ denote the center of $\bar{P}$ and let T be the preimage of $\bar{T}$ in P. Then T is elementary*, $T=\Omega_1(P)$, $T=Z(P)$ *and $T^{\#}$ is fused in $N_G(P)$.*

Clearly $\langle z^G\cap P\rangle\leqq T$ since $\bar{T}=\Omega_1(\bar{P})$. But since $N_{\bar{L}}(\bar{P})$ transitively conjugates the $q-1$ elements of $\bar{T}$ and some z^g lies in $T-V$ and $V=\langle z\rangle$, we have

$$T = \langle z^G \cap P\rangle.$$

Then $T\leqq Z(P)$. Since $\bar{T}=Z(\bar{P})$, $T=Z(P)$. Since T is generated by mutually commuting involutions it is elementary, hence $T\leqq\Omega_1(P)$. But $\bar{T}=\Omega_1(\bar{P})$ so $T=\Omega_1(P)$. Now T is elementary of order $2q$ and its nonzero elements are permuted by a subgroup of L into three orbits of lengths $1, q-1$, and $q-1$. Here $\{z\}$ comprises the orbit of length one, and one of the other orbits contains z^g. But since $z^G\cap P$ (being a subset of the center of P) is fused in $N_G(P)$ we see that $|z^G\cap P|$ is odd and so $z^G\cap P=T^{\#}$ This set is fused in $N_G(P)$.

(xiii) *Contradiction.*

Suppose $T\neq P$. Then $\bar{L}\simeq U(3, q)$ or $Sz(q)$. Suppose $z=w^2$ for some $w\in P$. Then $\bar{w}=wS/S$ is an involution in $\bar{P}$ and so lies in $\Omega_1(\bar{P})=Z(\bar{P})=\bar{T}$. But then $w\in T$ and so $w^2=1$ by (xii), a contradiction. Thus z is not a square in P. Then since $T^{\#}$ is fused in $N_G(P)$, no element of T is a square. But this is clearly impossible since any element of $P-T$ has order 4.

Thus $T=P$. Now P is abelian. Since P is a 2-Sylow subgroup of G this contradicts (ii).

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