PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 39, Number 1, June 1973

THE LARGEST SUM-FREE SUBSEQUENCE FROM A SEQUENCE OF n NUMBERS

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Abstract. Let $g(n)$ denote the largest integer so that from any sequence of n real numbers one can always select a sum-free subsequence of $g(n)$ numbers. Erdös has shown that $g(n)>2^{-1/2}n^{1/2}$. In this paper we obtain an improved estimate by a different method.

1. For any natural number n, let $g(n)$ denote the largest integer so that from any sequence of n real numbers one can always select a sum-free subsequence of $g(n)$ numbers, namely, a subsequence of $g(n)$ numbers none of which is the sum of other numbers of the subsequence. An estimate of the type

$$g(n) > cn^{1/2}, \qquad n \geqq n_0 \tag{1}$$

was first obtained by Erdös [1] with $c=2^{-1/2}$ and $n_0=1$. The purpose of this note is to establish (1) with $c=36/35$ and n_0 sufficiently large by an entirely different method. We mention that a slight improvement over 36/35 is certainly possible by our method but it appears that a new idea may be needed to improve the constant to 2.[1]

First, for the sake of comparison with our method, we recall Erdös' simple and elegant proof of (1) with $c=2^{-1/2}$.

Let $a_1, \cdots, a_n$ be a sequence of n numbers and T be large. Denote by $\mathscr{I}_j$ the set in α, $0<\alpha<T$ for which $a_j\alpha$ (mod 1) is between $(2n)^{-1/2}$ and $2^{1/2}n^{-1/2}$, and by $m(\mathscr{I}_j)$ the measure of $\mathscr{I}_j$. Then

$$\left| m(\mathscr{I}_j) - \frac{T}{(2n)^{1/2}} \right| < A$$

where A depends at most on the a's. From this inequality it follows that there is an α and at least $(n/2)^{1/2}$ a's, say a_{i_j} $(1\leqq j\leqq (n/2)^{1/2})$ such that $a_{i_j}\alpha$ (mod 1) is between $(2n)^{-1/2}$ and $2^{1/2}n^{-1/2}$.

Received by the editors July 6, 1972.
AMS (MOS) subject classifications (1970). Primary 10L10.
Key words and phrases. Sum-free, subsequence, real numbers.

[1] It has recently been brought to my knowledge that Professor Cantor has (essentially) improved the constant to $\sqrt{2}$ by a refinement of our method.

2. A very simple argument suffices to prove (1) with $c=1$. Let

$$a_1 \geqq a_2 \geqq \cdots \geqq a_n \tag{2}$$

be a sequence of n real numbers. We select from the sequence (2) a set of k distinct numbers $b_1, b_2, \cdots, b_k$ satisfying

$$b_i \geqq 2b_{i+1} \qquad (i = 1, 2, \cdots, k-1) \tag{3}$$

as follows. Let $b_1=a_1$ and suppose b_i $(i\geqq 1)$ has been chosen. Then we choose b_{i+1} to be the largest number from (2) not exceeding $\frac{1}{2}b_i$. Suppose b_k is the last number that can be so selected. The sequence $b_1>b_2>\cdots> b_k$ is clearly sum-free and we have (1) with $c=1$ if $k\geqq n^{1/2}$. Suppose now $k<n^{1/2}$. Then on putting $b_{k+1}=a_n-1$, we see that there exists some j^* $(1\leqq j^*\leqq k)$ such that at least $n^{1/2}$ numbers of (2) lie in the interval $(b_{j^*+1}, b_{j^*}]$. These numbers are sum-free since $b_{j^*}\leqq 2b_{j^*+1}+1$.

3. We proceed to obtain (1) with $c=36/35$. We let the numbers $b_1, \cdots, b_k$ be defined as in §2. We may clearly assume

$$k \leqq (36/35)n^{1/2} \quad \text{and} \tag{4}$$

$$|\mathcal{S}_j| \leqq (36/35)n^{1/2} \qquad (j = 1, \cdots, k) \tag{5}$$

where $\mathcal{S}_j$ denotes the set consisting of all numbers (not necessarily distinct) of (2) in the interval $(b_{j+1}, b_j]$, and where $|\mathcal{S}_j|$ denotes the number of elements of $\mathcal{S}_j$.

Let $\mathcal{S}_{i_j}$ $(j=1, \cdots, h)$ denote those sets containing at least $(31/35)n^{1/2}$ elements. We have, on using (5)

$$(k-h)(31/35n)^{1/2} + h(36/35)n^{1/2} \geqq n, \quad \text{i.e.} \quad h \geqq 7(n^{1/2} - (31/35)k)$$

which gives, because of (4)

$$h \geqq 7\left(1 - \frac{31.36}{35^2}\right)n^{1/2}. \tag{6}$$

Let h_1 be defined by

$$h - h_1 - 1 > \frac{\log((36/35)n^{1/2})}{\log 2} \geqq h - h_1 - 2. \tag{7}$$

We remark for future reference that a consequence of (7) is that every number in $\bigcup_{j=1}^{h_1} \mathcal{S}_{i_j}$ is larger than β which denotes the sum of all the numbers in $\mathcal{S}_{i_h}$. We also note that in view of (6) and (7) we have

$$h_1 \geqq (3/5)n^{1/2} \tag{8}$$

if n is sufficiently large.

We now consider two cases.

Case 1. There exists some j^* $(1 \leq j^* \leq h_1)$ such that $\mathcal{S}_{i_{j^*}}$ contains a set $\mathcal{U}$ of $\geq |\mathcal{S}_{i_{j^*}}|/6$ distinct numbers differing from one another by more than β.

Case 2. For each $j=1, 2, \cdots, h_1$, there exists in $\mathcal{S}_{i_j}$ a set $\mathcal{J}_j$ of 7 numbers all contained in an interval of length β.

If Case 1 holds then the set consisting of all the numbers in $\mathcal{U}$ and $\mathcal{S}_{i_h}$ is sum-free. To see this we first note that $\mathcal{U}$ and $\mathcal{S}_{i_h}$ are both sum-free. Next we recall that by construction any two numbers in $\mathcal{U}$ differ by more than β, which is the sum of all numbers in $\mathcal{S}_{i_h}$. Finally the remark following (7) completes the argument.

We next consider Case 2. We define v by

$$(9) \qquad \begin{aligned} v &= h_1/4 && \text{if 4 divides } h_1, \\ v &= [h_1/4] + 1 && \text{otherwise} \end{aligned}$$

and consider the v sets $\mathcal{J}_{1+4j}$ $(j=0, 1, \cdots, v-1)$. We shall show that $\bigcup_{j=0}^{v-1} \mathcal{J}_{i+4j}$ is sum-free. To this end it suffices to show

(i) For each i $(i=0, 1, \cdots, v-2)$ the sum of all the numbers in $\bigcup_{j=i+1}^{v-1} \mathcal{J}_{1+4j}$ is less than any number in $\mathcal{J}_{1+4i}$.

(ii) The difference between any two distinct numbers belonging to the same $\mathcal{J}_{1+4i}$ is less than any number in $\bigcup_{j=0}^{v-1} \mathcal{J}_{1+4j}$.

To see that (i) holds we note that the sum of all the numbers in $\bigcup_{j=i+1}^{v-1} \mathcal{J}_{1+4j}$ is $<7M_i(1+2(1/2^4+(1/2^4)^2+\cdots))<8M_i$ where M_i denotes largest number in $\mathcal{J}_{1+4(i+1)}$, whereas the smallest number in $\mathcal{J}_{1+4i}$ is at least $8M_i$.

For the purpose of establishing (ii) we recall that, by construction, two distinct numbers in the same $\mathcal{J}_{1+4j}$ differ by at most β whereas any number in $\bigcup_{j=0}^{v=1} \mathcal{J}_{1+4j}$ is larger than β, by the remark following (7).

Finally we see that Case 1 gives a sum-free set of cardinality at least

$$|\mathcal{U}| + |\mathcal{S}_{i_h}| > \frac{7}{6}\left(\frac{31}{35}\right) n^{1/2} = \frac{31}{30} n^{1/2},$$

and that Case 2 gives a sum-free set of cardinality at least

$$\left(\frac{h_1}{4}\right) 7 > \frac{7}{4} \cdot \frac{3}{5} n^{1/2} = \frac{21}{20} n^{1/2},$$

on using (9) and (8).

In either case (1) is satisfied with $c=36/35$.

Reference

1. P. Erdös, *Extremal problems in number theory*, Proc. Sympos. Pure Math., vol. 8, Amer. Math. Soc., Providence, R.I., 1965, pp. 181–189. MR **30** #4740.

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