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ON SOME FUNDAMENTAL PROBLEMS IN CLUSTER SET THEORY

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ABSTRACT. An attempt to characterize the local behavior of arbitrary functions of two real variables in terms of their cluster sets along various approach curves leads to two main problems: (1) Finding a "small" family Γ_0 of approach curves such that cluster sets along Γ_0 determine total cluster sets; (2) Finding a "large" family of approach curves along which cluster sets can be preassigned. A nice solution of (1) is found and (2) is partially solved. Conjectures are made concerning a link between (2) and sets which are always of the first category.

1. **Introduction.** If f is a real-valued function defined on the real line R, the sets $C^-(f)$ and $C^+(f)$ of left- and right-hand limit points of f at 0 (say) are closed. Conversely, if C^- and C^+ are closed sets, Bettazzi [**1**, p. 173] constructs a function f such that $C^-(f)=C^-$ and $C^+(f)=C^+$ (for a proof in English, see [**4**, p. 319]). Thus the local behavior of f is more or less characterized by pairs of closed sets. An attempt to extend this characterization to the two-dimensional case leads to these problems:

I. Find a "small" subset Γ_0 of the set Γ of all approach curves such that for each function f, $\{C(f,\gamma_0):\gamma_0\in\Gamma_0\}$ determines $C(f,\gamma)$ for all γ on Γ.

II. Find a "large" subset Γ_0 of Γ such that to any collection $\{C(\gamma_0):\gamma_0\in\Gamma_0\}$ of closed sets there corresponds a function f such that $C(f,\gamma_0)=C(\gamma_0)$ for all γ_0 in Γ_0.

Here $C(f,\gamma)$ denotes the *cluster set* of f at 0 along the approach curve γ, i.e., $C(f,\gamma)=\{\lambda$: there exists a sequence $\{P_n\}$ of distinct points on γ such that $P_n\to 0$ and $f(P_n)\to\lambda\}$. (For the general theory of cluster sets, see [**2**] and [**6**].) By an *approach curve* will be meant the graph or any rotation of the graph of a real-valued continuous function on $[0,\infty)$. Adjectives "convex", "continuously differentiable", etc., when applied to approach curves are defined in terms of the reference function. Thus, a convex approach curve is the graph or rotation of the graph of a convex continuous function on $[0,\infty)$. (A straight line is regarded as a special case of

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a convex approach curve.) Two approach curves will be identified if they coincide in some neighborhood of the origin.

Ideally, Problems I and II would have a common solution. However, we will see that this is asking for too much.

2. **Solution of Problem I.** In a strict sense, Problem I has no solution. However, by properly interpreting the word "determines" we are able to get a most satisfying solution.

THEOREM 1. *If Γ_0 is a proper subset of Γ and γ is an element of $\Gamma-\Gamma_0$, then there exist functions f and g such that $C(f,\gamma_0)=C(g,\gamma_0)$ for every γ_0 in Γ_0 but $C(f,\gamma)\neq C(g,\gamma)$.*

PROOF. Let (A, B) be a decomposition of the plane ($A\cup B=R^2$, $A\cap B=\varnothing$) such that A and B contain no arcs. (Proof that such a decomposition exists: Well order the set of all arcs in such a way that any initial segment of the ordering has cardinality less than the cardinality of the continuum; then, observing that the set of all arcs of the plane is equivalent to the set of all points of the plane, construct disjoint sets A_0 and B_0 inductively so that both A_0 and B_0 intersect every arc; finally, set $A=A_0$ and $B=R^2-A$.) For $P\notin\gamma$ define

$$\begin{aligned} f(P) = g(P) &= 0 \quad \text{if } P\in A,\\ &= 1 \quad \text{if } P\in B \end{aligned}$$

and for $P\in\gamma$ define $f(P)=0$ and $g(P)=1$. Then $C(f,\gamma_0)=\{0,1\}=C(g,\gamma_0)$ for all γ_0 in Γ_0 but $C(f,\gamma)=\{0\}$ and $C(g,\gamma)=\{1\}$. ∎

Henceforth, we will interpret "determines" in Problem I to mean only that

$$(*)\qquad C(f,\gamma)\subset\bigcup_{\gamma_0\in\Gamma_0}C(f,\gamma_0)$$

for every γ in Γ.

THEOREM 2. *If Γ_0 is the set of all convex differentiable approach curves, then $(*)$ holds for any function f and any $\gamma\in\Gamma$. However, if Γ_0 is only the set of all (at least) twice differentiable approach curves, then $(*)$ fails for some function f and some γ in Γ.*

The result is sharp because any convex differentiable approach curve is continuously differentiable (see, e.g., [3]).

LEMMA. *If $\{P_n\}$ is any sequence of distinct points converging to 0, then there exists a convex differentiable approach curve γ_0 and a subsequence $\{P_{n_k}\}$ of $\{P_n\}$ such that $P_{n_k}\in\gamma_0$ for all k. On the other hand, the approach curve γ defined by $y=x^{3/2}$, $x\geqq 0$, has the property that there is no twice* *differentiable approach curve which intersects* γ *on a sequence* $\{P_n\}$ *of distinct points converging to* 0.

The lemma is a modification of a result due to A. Rosenthal [7].

PROOF OF LEMMA. The first statement is equivalent to the first part of Theorem 2 of [7]. For the second statement, let γ_0 be a differentiable approach curve which intersects γ on an infinite sequence of points converging to 0. We show that γ_0 cannot be twice differentiable by showing that it cannot have a second derivative at 0.

Even though γ_0 contains a sequence of points converging to 0 in the first quadrant, it may happen that it is impossible to represent it by a single-valued function. However, if γ_0 is rotated through a suitable acute angle θ, it can be represented in some positive half-neighborhood of 0 by a differentiable function $\phi_0(x)$. Moreover, when γ is rotated through the same angle, it can be represented in some positive half-neighborhood of 0 by a function $\phi(x)$ having these properties:

(i) $\phi(0)=0$, $\phi'(0)=\tan\theta$,

(ii) $(\phi'(x)-\tan\theta)/x\to+\infty$ as $x\downarrow 0$,

(iii) $\phi(x_n)=\phi_0(x_n)$ for some sequence $x_n\downarrow 0$.

By (i) and (iii), $\phi_0'(0)=\tan\theta$ and it follows from (ii) that $[\phi_0'(x)-\tan\theta]/x$ cannot be bounded. (If it were, we would have $\phi'(x)>\phi_0'(x)$ and hence $\phi(x)>\phi_0(x)$ in some positive half-neighborhood of 0 contrary to (iii).) Hence, $\phi_0''(0)$ cannot exist. (Cf. the proof of Theorem 2 of [7].) ∎

PROOF OF THEOREM 2. If $\lambda\in C(f,\gamma)$, then there exists a sequence $\{P_n\}$ of distinct points on γ such that $P_n\to 0$ and $f(P_n)\to\lambda$. It then follows from the lemma that $\lambda\in C(f,\gamma_0)$ for some convex differentiable approach curve γ_0.

For the second part, let γ be as in the lemma and define $f(P)=1$ for $P\in\gamma$, $f(P)=0$ for $P\notin\gamma$. Then $C(f,\gamma)=\{1\}$ and, by the lemma, $C(f,\gamma_0)=\{0\}$ for every twice differentiable approach curve γ_0. ∎

3. **Partial solution of Problem II.** We start by looking at some simple examples. First, Bettazzi's construction shows that any one point family $\Gamma_0=\{\gamma_0\}$ is a solution. Indeed, the same reasoning shows that any non-intersecting family such as the family of all radial approaches is a solution. It is also clear that any family that intersects itself at only a finite number of points is a solution. However, a family that intersects itself infinitely many times need not be a solution. This is a consequence of Theorem 3 (below).

Two approach curves will be called *comparable* if they are disjoint in some deleted neighborhood of the origin. A family Γ_0 of approach curves will be called comparable if any two curves in Γ_0 are comparable.

THEOREM 3. *Any family which is a solution of Problem* II *must be comparable.*

It follows that there exist families of convex differentiable approach curves which are not solutions of Problem II. In fact, if $\psi(x)=(\sin \pi/x)e^{-1/x}$ for $x>0$, $\psi(0)=0$, and M is a bound for ψ'' on $[0, \infty)$, then the approach curves γ_1 and γ_2 defined for $x\geqq 0$ by $y_1(x)=Mx^2$ and $y_2(x)=Mx^2+\psi(x)$, respectively, are noncomparable, infinitely-differentiable, and convex.

PROOF OF THEOREM 3. If γ_1 and γ_2 are noncomparable approach curves, they intersect on a sequence $\{P_n\}$ of distinct points converging to 0. Hence, $C(f, \gamma_1)\cap C(f, \gamma_2)\neq\varnothing$ for every function f. ∎

THEOREM 4. *A comparable family* Γ_0 *of approach curves is a solution of Problem* II *if either* Γ_0 *is countable or else, in the uncountable case, intersects itself only a countable number of times.*

PROOF. If $\Gamma_0=\{\gamma_1, \gamma_2, \cdots\}$, proceed inductively defining f on $\gamma_{N+1}-\bigcup_{n=1}^N \gamma_n$ so that $C(f, \gamma_{N+1})=C(\gamma_{N+1})$. This is possible because γ_{N+1} and $\bigcup_{n=1}^N \gamma_n$ will have an empty intersection in some deleted neighborhood of the origin. The function f may be defined arbitrarily off Γ_0.

If Γ_0 intersects itself countably-many times, we can write $\Gamma_0=\Gamma_1\cup\Gamma_2$ where Γ_1 is countable and Γ_2 is nonintersecting. Since Γ_1 and Γ_2 are solutions of Problem II and since there are no intersections between these families, Γ_0 is also a solution. ∎

If a comparable family intersects itself uncountably-many times, it may or may not be a solution of Problem II.

EXAMPLE 1. $\{y=ax^2:a\in R\}\cup\{y=mx:m\in R\}$ is a solution.

PROOF. The curve $y=x^{3/2}$ serves as a "dividing curve" (in the first quadrant) in the sense that all of the lines are eventually above it and all of the parabolas are eventually below it. Bettazzi's construction can now be applied. ∎

EXAMPLE 2. $\{y=ax+x^2:a\in R\}\cup\{y=bx-x^2:b\in R\}$ is not a solution.

PROOF. If the given family were a solution, there would exist a function f with limiting value 0 along curves of the form $y=ax+x^2$ and limiting value 1 along curves of the form $y=bx-x^2$. Hence there would exist positive functions $r(a)$ and $s(b)$ such that $f(x, ax+x^2)<\frac{1}{2}$ whenever $x<r(a)$ and $f(x, bx-x^2)>\frac{1}{2}$ whenever $x<s(b)$. It would then follow that each of the sets $A_n=\{a\in R:r(a)>1/n\}$ is nowhere dense and hence that R was of the first category. ∎

4. **Unsolved problems, conjectures.** By a *truncation* of an approach curve γ will be meant the restriction of γ to some deleted neighborhood

of 0. If every curve of a given family is truncated, the corresponding family of truncations is called a truncation of the original family. In the proofs for the above examples, we were essentially showing the existence or nonexistence of nonintersecting truncations.

CONJECTURE. A family Γ_0 is a solution of Problem II if and only if Γ_0 has a nonintersecting truncation.

The nonintersecting truncation problem is related to and dependent upon an even more basic problem. Consider a set S of real numbers and a real-valued function m defined on S. For $s \in S$, let L_s denote the half-line $y=m(s)x+s$, $x \geqq 0$, and let $\mathscr{L}=\{L_s : s \in S\}$. Call (S, m) *acceptable* if $\mathscr{L}$ has a nonintersecting truncation. ("Truncation" is defined in exactly the same way as before.) Nongeometrically, this corresponds to the existence of a positive function p on S such that

$$\frac{m(s)-m(t)}{t-s} < \max(p(s), p(t))$$

whenever $s, t \in S$, $s \neq t$.

CONJECTURES. (1) (S, m) is acceptable if and only if for each decomposition (A, B) of S there exists a positive function p on S such that

$$\frac{m(a)-m(b)}{b-a} < \max(p(a), p(b))$$

whenever $a \in A$, $b \in B$; (2) (S, m) is acceptable for all m if and only if S is a set which is *always of the first category* [**5**, p. 516]; (3) S is always of the first category if and only if for each decomposition (A, B) of S there exist sequences of sets $\{A_n\}$ and $\{B_n\}$ such that

(i) $A_1 \subset A_2 \subset \cdots$, $B_1 \subset B_2 \subset \cdots$,

(ii) $A=\bigcup_n A_n$, $B=\bigcup_n B_n$, and

(iii) $\bar{A}_n \cap B_n = \varnothing$, $A_n \cap \bar{B}_n = \varnothing$.

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