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C^k, WEAKLY HOLOMORPHIC FUNCTIONS ON ANALYTIC SETS[1]

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ABSTRACT. Let V be a complex analytic set and $p \in V$. Let $\mathcal{O}(V)$, $\tilde{\mathcal{O}}(V)$, and $C^k(V)$ denote respectively the rings of germs of holomorphic, weakly holomorphic, and k-times continuously differentiable functions on V. Spallek proved that there exists sufficiently large k such that $C^k(V) \cap \tilde{\mathcal{O}}(V) = \mathcal{O}(V)$. In this paper I give a new proof of this result for curves and hypersurfaces which also establishes that the conduction number of the singularity is an upper bound for k. This estimate also holds for any pure dimensional variety off of a subvariety of codimension two.

Let V be a complex analytic set and $p \in V$. Let $\mathcal{O}(V)$, $\tilde{\mathcal{O}}(V)$, and $C^k(V)$ denote respectively the rings of germs of holomorphic, weakly holomorphic, and k-times continuously differentiable functions on V. In [**6**], Spallek proved that there exists sufficiently large k such that $C^k(V) \cap \tilde{\mathcal{O}}(V) = \mathcal{O}(V)$. In this paper I give a new proof of this result for curves and hypersurfaces which also establishes that the conduction number of the singularity is an upper bound for k. This estimate also holds for any pure dimensional variety off of a subvariety of codimension two.

An element $u \in \mathcal{O}$ is said to be a universal denominator if $u\tilde{\mathcal{O}} \subset \mathcal{O}$. Let I be the ideal of $\mathcal{O}$ of all functions vanishing on Sing(V) and J be the ideal of universal denominators. Then locus(J) $\subset$ Sing V [**3**, p. 56], so by the Hilbert Nullstellensatz there is a positive integer N such that $I^N \subset J$. The main result of this paper is that k can be chosen so that $k \leq N$.

Siu has proven [**5**] that if $k(p)$ is the minimal value of k such that $C^k \cap \tilde{\mathcal{O}} = \mathcal{O}$ at the point $p \in V$, then the function $k(p)$ is bounded on compact subsets of V. This result also follows from the above estimate, by the coherence of the ideal sheafs of I and J. (The ideal sheaf of J is coherent [**2**, Theorem 22] because it is the kernel of $\mathcal{O} \to \mathrm{Hom}_{\mathcal{O}}(\tilde{\mathcal{O}}, \tilde{\mathcal{O}}/\mathcal{O})$.)

This estimate is, in general, not the best possible. In an earlier work [**1**], for the example of a curve in $\mathbf{C}^2$ normalized by a map $t \to (t^q, t^p u(t))$ where $u(t)$ is a unit, $p > q$, and p and q are relatively prime, it was shown that

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$k=[(p/q)(q-2)]+1$ and $N=[(p/q)(q-1)]$, where $[x]$ for any real number x is the greatest integer less than or equal to x.

The original estimate for k obtained by Spallek **[6]** seems a bit obscure in the case of a nonisolated singularity. This is made clearer by Siu in **[5]** and Spallek in **[8]**:

Suppose A is an analytic set of pure dimension r, $\pi: A \to \boldsymbol{C}^r$ a branched covering of sheeting order μ, z_{r+1} a direction in $\boldsymbol{C}^n$ which separates the fibers of π almost everywhere, and δ the discriminant of the minimal polynomial in ${}_A\mathcal{O}$ for z_{r+1} over ${}_r\mathcal{O}$; then $k \leqq \mu(m+1)$ where $I(\operatorname{locus}(\delta))^m \subset \delta_A \mathcal{O}$. Now m is related to the conduction number N, but not necessarily equal—depending upon whether the projection π has minimal multiplicity—so the estimate in this paper is better approximately by a factor of the minimal multiplicity. For the above mentioned case of a curve in $\boldsymbol{C}^2$, direct computation shows that Spallek's estimate is $k \leqq p(q-1)+q$.

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1. Suppose V is a complex analytic hypersurface in $\boldsymbol{C}^n$, the projection $\pi: \boldsymbol{C}^n \to \boldsymbol{C}^{n-1}$ to the first $n-1$ coordinates gives a q sheeted branched cover of V with branch set B, $B'=\pi(B)$ and $z'=\pi(z)$. Now π induces a homomorphism ${}_{n-1}\mathcal{O} \to {}_n\mathcal{O}/I(V)=\mathcal{O}(V)$ making $\mathcal{O}(V)$ into a finitely generated ${}_{n-1}\mathcal{O}$ module with generators $1, z_n, \cdots, z_n^{q-1}$. Hence if $f \in \mathcal{O}(V)$, then f can be written as $\sum_{i=0}^{q-1} b_i(z') z_n^{q-i-1}$. For any weakly holomorphic function f, there is a canonical attempted extension to the ambient space, which is in fact holomorphic, if f is holomorphic: for $z' \notin B'$, let

$$g(z', z_n) = \sum_{j=1}^{q} \left(\prod_{k \neq j} \frac{z_n - \alpha_k(z')}{\alpha_j(z') - \alpha_k(z')} \right) f(z', \alpha_j(z'))$$

$$= \sum_{i=0}^{q-1} (-1)^i b_i(z') z_n^{q-i-1},$$

$$b_i(z') = \sum_{j=1}^{q} \frac{\sigma_i(\alpha_1(z'), \cdots, \hat{\alpha}_j(z'), \cdots, \alpha_q(z'))}{\prod_{k \neq j} (\alpha_j(z') - \alpha_k(z'))} f(z', \alpha_j(z'))$$

where hatted terms are deleted, σ_i is the elementary symmetric polynomial of degree i, and $\{\alpha_j(z'): 1 \leqq j \leqq q\}$ are the values of z_n on the fiber $\pi^{-1}(z')$.

For $z \in V$, $z_n=$some α_j so $g(z) \equiv f(z)$. The coefficients b_i are well defined (do not depend upon the ordering of the α_i's), $b_i \in \mathcal{O}(\boldsymbol{C}^r - B')$ and $g \in \mathcal{O}(\boldsymbol{C}^n - B' \times \boldsymbol{C}^{n-r})$. By the Riemann removable singularities theorem, g extends holomorphically to $\boldsymbol{C}^n$ if and only if the b_i are bounded near B'. (If g is holomorphic, then so is $(\partial^{q-1}/\partial z_n) g = (q-1)!\, b_0(z')$, etc.)

If $f \in \mathcal{O}(V)$, then as pointed out by Spallek [5, Abschnitt 6], the Newton Interpolation Formula [4, pp. 10–16] says that if

$$[f_1, \cdots, f_q] = \sum_{j=1}^{q} \frac{f(z', \alpha_j(z'))}{\prod_{j \neq k} (\alpha_j(z') - \alpha_k(z'))},$$

then there exists a complex constant λ with $|\lambda| \leqq 1$ and real numbers $\delta_1, \cdots, \delta_q \geqq 0$ with $\sum \delta_i = 1$ such that

$$[f_1, \cdots, f_q] = (\lambda/(q-1)!)(\partial^{q-1}/\partial z_n) f(z', \delta_1 \alpha_1 + \cdots + \delta_q \alpha_q).$$

Now

$$\begin{aligned} \sigma_i(\alpha_1, \cdots, \hat{\alpha}_j, \cdots, \alpha_q) &= \sigma_i(\alpha_1, \cdots, \alpha_q) - \alpha_j \sigma_{i-1}(\alpha_1, \cdots, \hat{\alpha}_j, \cdots, \alpha_q) \\ &= \sum_{l=0}^{i} (-1)^l \alpha_j^l \sigma_{i-l}(\alpha_1, \cdots, \alpha_q) \end{aligned}$$

so

$$b_i(z') = \sum_{l=0}^{i} (-1)^l \sigma_{i-l}(\alpha_1, \cdots, \alpha_q)[(z_n^l f)_1, \cdots, (z_n^l f)_q]$$

and it follows immediately that b_i is bounded near B' by the continuity of $(\partial^{q-1}/\partial z_n) f$.

2. First we consider the one-dimensional case. Let V be normalized by a map $\theta(t) = (t^q, t^{p_1} u_1(t), \cdots, t^{p_{n-1}} u_{n-1}(t))$ where each $p_i \geqq q$ and each u_i is a holomorphic function with $u_i(0) \neq 0$. Let $f \in C^k(V) \cap \tilde{\mathcal{O}}(V)$ and $T_0^k(f)$ be the kth order Taylor series of f about the origin; write $T_0^k f = P_k f + Q_k f$ where $P_k f$ is a homomorphic polynomial and $Q_k f$ contains the antiholomorphic terms. It is a standard fact that $f - T_0^k f = o(|z|^k)$. However even more is true:

LEMMA. $(f - P_k f)\theta(t) = o(\theta(t)^k) = o(t^{qk})$.

This is Lemma 3 of [1] and is also essentially contained in [5, paragraph 2.2].

Let $h = f - P_k f$; we have that h is also weakly holomorphic, h is holomorphic if and only if f is holomorphic, h is precisely as differentiable as f and that $h(\theta(t)) = o(t^{qk})$ since $P_k h \equiv 0$. Hence h/z_1^k is weakly holomorphic. Since z_1^N is a universal denominator, $h z_1^{N-k}$ is holomorphic; for $k = N$ we have that h is holomorphic. Thus $C^N(V) \cap \tilde{\mathcal{O}}(V) = \mathcal{O}(V)$.

More generally, for a variety V of pure dim r in C^n, let

$$\begin{aligned} C = {} & \operatorname{Sing}(\operatorname{Sing} V) \\ & \cup \{p \in V \mid \dim C_4(V, p) > r\} \cup \{p \in V \mid \dim C_5(V, p) > r + 1\} \end{aligned}$$

where $C_4(V, p)$ and $C_5(V, p)$ are the fourth and fifth Whitney tangent cones to V at p [**10**]. Then C is an analytic subset of V of codimension at least two [**9**, Proposition 3.6] and every $p \in V - C$ has an open neighborhood so that after a local biholomorphic change of coordinates the following hold:

(i) For each irreducible component V_i of V, $V_i \cap \operatorname{Sing} V = \operatorname{Sing} V_i = \boldsymbol{C}^{r-1}$ [**9**, Proposition 2.10, 2.12, and 4.5].

(ii) Each component has a one-to-one nonsingular normalization [**9**, Proposition 4.2] $\phi: D \to V_i$ given by

$$\phi(t_1, \cdots, t_r) = (t_1, \cdots, t_{r-1}, t_r^q, \phi_{r+1}(t), \cdots, \phi_n(t)),$$

where q is the sheeting order of $\pi | V_i$ and $\pi(x_1, \cdots, x_n) = (x_1, \cdots, x_r)$. The branching set of this projection is just $\phi(\{t_r = 0\}) = \boldsymbol{C}^{r-1}$.

Now let $\operatorname{Cond}_p(V)$ denote the conduction number of the variety at the point p as defined in the introduction. If V_i is a component of V it is clear that any universal denominator for V is a universal denominator for V_i and since $\operatorname{Sing} V_i = \operatorname{Sing} V$, we have that $\operatorname{Cond}_p(V) \geqq \operatorname{Cond}_p(V_i)$.

For any fixed $s = (t_1, \cdots, t_{r-1})$ consider the curve W_s in V_i given by $t_r \to \phi(s, t_r)$. Since this curve W_s lies in $s \times \boldsymbol{C}^{n-r+1}$, weakly holomorphic functions on W_s extend to weakly holomorphic functions on V_i by ignoring the first $r-1$ variables. Hence any universal denominator for V_i is a universal denominator for W_s and $\operatorname{Cond}_p(V_i) \geqq \operatorname{Cond}_p(W_s)$.

Suppose $f \in C^k \cap \tilde{\mathcal{O}}$, $k \geqq \operatorname{Cond}_p(V)$, and $r = n-1$; recall the canonical extension of §1, $\sum b_i(z') z_n^{q-i-1}$, with $b_i \in \mathcal{O}(\boldsymbol{C}^r - \boldsymbol{C}^{r-1})$. Since W_s is a hypersurface in $s \times \boldsymbol{C}^2$ and $k \geqq \operatorname{Cond}_{\phi(s,0)}(W_s)$ for each s, by the one-dimensional case we have that b_i is bounded on each line

$$L_s = \{(s, z_r) : z_r \in \boldsymbol{C}\}.$$

We need to conclude that b_i extends holomorphically to $\boldsymbol{C}^r$. To do this consider the Laurent power series expansion of b_i in $\boldsymbol{C}^r - \boldsymbol{C}^{r-1}$:

$$b_i(z') = \sum_{m=-\infty}^{+\infty} a_m(z_1, \cdots, z_{r-1}) z_r^m, \qquad a_m \in \mathcal{O}(\boldsymbol{C}^{r-1}),$$

$$a_m(z'') = \frac{1}{2\pi i} \oint \frac{b_i(z'', \xi)\, d\xi}{\xi^{m+1}}.$$

Choose an $s \in \boldsymbol{C}^{r-1}$ such that for each a_m which is not identically zero, $a_m(s) \neq 0$; if there are any negative exponents of z_r in the above power series expansion, b_i is not bounded on the line L_s (neither a pole nor an essential singularity is bounded).

So far we have shown that $k \leqq \operatorname{Cond}_p(V)$ for $p \in V - C$. By coherence of the ideal sheafs of $I(\operatorname{Sing} V)$ and J, if $c \in C$, then $\operatorname{Cond}_c(V) \geqq \operatorname{Cond}_p(V)$

for all p near c. If $f \in C^k \cap \tilde{\mathcal{O}}$ with $k \geqq \mathrm{Cond}_c(V)$ then by the last paragraph $b_i \in \mathcal{O}(C^r - \pi(C))$; but dim $\pi(C) \leqq r-2$ so by Hartog's theorem [**3**, p. 59], $b_i \in \mathcal{O}(C^r)$ and $f \in \mathcal{O}(V)$.

3. Even without the assumption about the codimension of V, it follows that for pure r-dimensional V there exists an analytic subset W of V with dim $W \leqq r-2$ such that for every $p \in V-W$, $k_{(p)} \leqq N_{(p)}$. Of course this implies the result of the last section since for algebraic complete intersections, singularities in codim two are removable [**11**].

Instead of directly exhibiting the holomorphic extension, we must resort to more delicate results in sheaf theory, due to Spallek [**8**, Satz 3.2].

If f is a weakly holomorphic function on V, $1 \leqq q \leqq r$, and f restricted to each q-dimensional parallel section is holomorphic, then there is an analytic subset B^q of V (not depending upon f) of dimension at most $r-q-1$ so that f is holomorphic on $V-B^q$.

It was shown in the previous section that by restricting to $V-C$, we have f holomorphic on each $s \times C^{n-r}$ and $q = \dim V \cap (s \times C^{n-r}) = 1$ so dim $B^1 \leqq r-2$ and f is holomorphic on $V-(C \cup B^1)$.

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