

ON THE GROWTH OF THE TAYLOR COEFFICIENTS OF AUTOMORPHIC FORMS

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ABSTRACT. The growth of the Taylor coefficients of an automorphic form of dimension -2 with respect to a Fuchsian group Γ is related to the area integral $\iint_U |F|^s(1-|z|^2)^t dx dy$, and it is found that these coefficients must grow faster than a power of n . Moreover if $F \in H(p, \Gamma)$ then these coefficients must grow slower than a different power of n and, in fact, a_n/n is square summable if either $p=2$ or $1 < p < \infty$ and Γ is finitely generated of the second kind.

1. Introduction. Throughout Γ shall stand for a Fuchsian group acting on the unit disk $U(=\{z:|z|<1\})$ of the complex plane. We shall assume that Γ is of convergence type, i.e.,

$$(1.1) \quad \sum_{T \in \Gamma} |T'(z)| < \infty \quad \text{for all } z \text{ in } U.$$

We note that (1.1) is equivalent to the fact that the associated Riemann surface U/Γ is hyperbolic (cf. Tsuji [9, p. 522]). If F is a function defined on U , then we say that F is an automorphic form of degree -2 if

$$F(Tz)T'(z) = F(z) \quad \text{for all } T \text{ in } \Gamma \text{ and } z \text{ in } U.$$

We define

$$(1.2) \quad \lambda^{-1}(z) = (1 - |z|^2), \quad d\omega(z) = \lambda^2(z) dx dy,$$

and note that $\lambda^{-1}(Tz) = |T'(z)|\lambda^{-1}(z)$ and $d\omega(Tz) = d\omega(z)$ for all conformal automorphisms T of U onto itself. If Ω is a fundamental region for Γ whose boundary has two dimensional measure zero, then we define $H(p, \Gamma)$ ($1 \leq p < \infty$) to be the space of holomorphic automorphic forms of

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degree -2 with respect to Γ such that $\|F\|_p < \infty$ where

$$\|F\|_p^p = \iint_{\Omega} |F(z)|^p \lambda^{-p}(z) d\omega(z).$$

We note that $H(1, \Gamma) = \{0\}$ for a large class of groups Γ (cf. [5, Theorem 3]). $H(\infty, \Gamma)$ is defined to be the space of holomorphic automorphic forms of degree -2 with respect to Γ with the norm

$$\|F\|_{\infty} = \sup_{z \in U} |F(z)| \lambda^{-1}(z) < \infty.$$

In this paper we shall obtain estimates on the growth of the Maclaurin coefficients of holomorphic automorphic forms of dimension -2 by relating such growth to the finiteness of integrals of the form (cf. [3])

$$(1.3) \quad \iint_U |F|^s (1 - |z|^2)^t dx dy.$$

Although all of our work is done for the case Γ of convergence type and automorphic forms of dimension -2 , analogous results could be stated and proved for arbitrary Γ and automorphic forms of dimension $-2q < -2$.

2. Lower limits on the order of growth of the coefficients. We shall first show that the Maclaurin coefficients of an automorphic form cannot grow "too" slowly.

THEOREM 1. *Suppose Γ is such that $\sum_{T \in \Gamma} |T'(z)|^r = \infty$ for some z in U . Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$, $F \neq 0$, be an arbitrary (holomorphic) automorphic form of degree -2 with respect to Γ , then, for any $t < r/2$, $a_n \neq O(n^t)$.*

Note. If $\sum_{T \in \Gamma} |T'(t)|^r = \infty$ for one z in U then $\sum_{T \in \Gamma} |T'(t)|^r = \infty$ for all z in U .

PROOF. Suppose $a_n = O(n^t)$, then $\sum_{n=0}^{\infty} (|a_n|^2 / (n+1)^{1+r}) < \infty$. Since $\Gamma(x)/\Gamma(x+a) \sim x^{-a}$, it follows that

$$\sum_{n=0}^{\infty} \frac{|a_n|^2 \Gamma(r+1)\Gamma(n+1)}{\Gamma(n+r+2)} < \infty,$$

this in turn implies, since $\beta(p, q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$,

$$\sum_{n=0}^{\infty} |a_n|^2 \int_0^1 (1-u)^r u^n du < \infty.$$

Now upon letting $u = \rho^2$, where $|z| = \rho$, we see that Parseval's formula

implies that

$$\iint_U |F|^2 (1 - |z|^2)^r dx dy = \sum_{n=0}^{\infty} |a_n|^2 \int_0^1 (1 - \rho^2)^r \rho^{2n+1} d\rho < \infty.$$

But this cannot happen if $F \neq 0$ for

$$\begin{aligned} \infty &> \iint_U |F|^2 (1 - |z|^2)^r dx dy = \sum_{T \in \Gamma} \iint_{T\Omega} |F|^2 (1 - |z|^2)^r dx dy \\ &= \iint_{\Omega} |F|^2 (1 - |z|^2)^r \sum_{T \in \Gamma} |T'(z)|^r dx dy, \end{aligned}$$

since $|F|^2 dx dy$ is Γ invariant. Since $\sum_{T \in \Gamma} |T'(z)|^r \equiv \infty$ we have arrived at a contradiction and the proof of the theorem is complete.

COROLLARY 2. *Suppose F is as above and Γ is not cyclic hyperbolic, then $a_n \neq 0(1)$.*

PROOF. If Γ is not cyclic hyperbolic, then Beardon [1] proved that there exists an $r > 0$ such that $\sum_{T \in \Gamma} |T'(t)|^r = \infty$; hence choosing $t=0$ in Theorem 1 completes the proof.

REMARK. If Γ is cyclic hyperbolic, Beardon [1, p. 475] showed that $\sum_{T \in \Gamma} |T'(t)|^r < \infty$ for all $r > 0$, however in this case we have $a_n \neq O(n^{-\epsilon})$ for all $\epsilon > 0$.

3. Upper limits on the growth of the coefficients. We note that in the above theorem and its corollary no assumption was made about the growth of F . If we make such an assumption, i.e., $F \in H(p, \Gamma)$, then it can be shown that the Maclaurin coefficients of F cannot grow "too" fast. In order to prove this we first need

LEMMA 3. *Let $F \in H(p, \Gamma)$ ($1 \leq p < \infty$). Then (1.3) is finite with $s=p$ and $t=p-1$.*

PROOF.

$$\begin{aligned} \iint_U |F|^p (1 - |z|^2)^{p-1} dx dy &= \iint_U |F\lambda^{-1}|^p (1 - |z|^2) d\omega(z) \\ &= \iint_{\Omega} |F\lambda^{-1}|^p \sum_{T \in \Gamma} (1 - |Tz|^2) d\omega(z) \\ &\leq \left[\sup_U \sum_{T \in \Gamma} (1 - |Tz|^2) \right] \|F\|_p^p. \end{aligned}$$

This last term is finite by Theorem 3 of [7] and the fact $F \in H(p, \Gamma)$, if Γ acts freely on U . If the origin is not a fixed point of Γ , then the proof of Theorem 3 of [7] goes through without modification to yield the desired result. If the origin is a fixed point of Γ , then, by conjugation, one can return to the previous case and again the result follows. This completes the proof of the lemma.

PROPOSITION 4. *Suppose $F(z) = \sum_{n=0}^{\infty} a_n z^n \in H(p, \Gamma)$ ($1 < p < \infty$). Then (1.3) is finite if $s=1$ and $t>0$; thus $a_n = O(n^{1+t})$ as $n \rightarrow \infty$.*

PROOF.

$$\begin{aligned} \iint_U |F| (1 - |z|^2)^t dx dy &= \iint_U |F| (1 - |z|^2)^{(p-1)/p} (1 - |z|^2)^{t-(p-1)/p} dx dy \\ &\leq \left(\iint_U |F|^p (1 - |z|^2)^{p-1} dx dy \right)^{1/p} \\ &\quad \left(\iint_U (1 - |z|^2)^{tp/(p-1)-1} dx dy \right)^{(p-1)/p} < \infty, \end{aligned}$$

by Hölder's inequality, Lemma 3, and the fact $t>0$. The conclusion that $a_n = O(n^{1+t})$ as $n \rightarrow \infty$ now follows from Theorem 4 of [3].

REMARK. J. Lehner (private communication) has shown that if

$$F \in H(p, \Gamma) \quad (1 \leq p < \infty), \quad \text{then } a_n = O(n).$$

4. Summability results. From Proposition 4 we saw that $a_n = O(n^{1+t})$ for all $t>0$ if $\sum_{n=0}^{\infty} a_n z^n \in H(p, \Gamma)$. However even more is true if we assume $2 \leq p < \infty$ for arbitrary Γ or $1 < p < \infty$ and Γ a finitely generated group. We note that if Γ is finitely generated then Γ is of convergence type if and only if Γ is of the second kind. We first prove

LEMMA 5. *If Γ is finitely generated of the second kind then, for $1 < p < \infty$,*

$$(4.1) \quad \iint_{\Omega} \left(\sum_{T \in \Gamma} 1 - |Tz|^2 \right)^p d\omega(z) < \infty.$$

The idea of proof is that if one chooses Ω to be a Dirichlet region, then, $\partial\Omega \cap \partial U$ consists of free sides and parabolic cusps. On the free sides $\sum_{T \in \Gamma} |T'(z)|$ has a bounded supremum and so we must integrate $(1 - |z|^2)^{p-2} dx dy$ which is finite. As for the parabolic cusps we need only recall that the hyperbolic area of a parabolic cusp is finite (cf. [4] for the details). Since $\|\cdot\|_p$ is independent of the choice made for the fundamental region the result follows.

THEOREM 6. *Suppose $F(z) = \sum_{n=0}^{\infty} a_n z^n$. Then $\{a_n/n\} \in l_2$ if either*

- (i) $F \in H(2, \Gamma)$ or
- (ii) $F \in H(p, \Gamma)$ ($1 < p < \infty$) and Γ is finitely generated.

PROOF. In view of Parseval's formula for functions holomorphic in U it suffices to show that (1.3) is finite with $s=2$ and $t=1$. Lemma 3 immediately yields the result in case (i). If Γ is finitely generated, then, for $1 < p \leq q \leq \infty$,

$$(4.2) \quad H(p, \Gamma) \subseteq H(q, \Gamma).$$

Assuming this for the moment, we see immediately that the theorem is true for $1 < p \leq 2$ in case (ii). If $p > 2$, then

$$\begin{aligned} \iint_U |F|^2 (1 - |z|^2) dx dy &= \iint_U |F\lambda^{-1}|^2 (1 - |z|^2) d\omega(z) \\ &= \iint_{\Omega} |F\lambda^{-1}|^2 \sum_{T \in \Gamma} (1 - |Tz|^2) d\omega(z) \\ &\leq \left(\iint_{\Omega} |F\lambda^{-1}|^p d\omega(z) \right)^{2/p} \\ &\quad \times \left(\iint_{\Omega} \left(\sum_{T \in \Gamma} (1 - |Tz|^2) \right)^{p/(p-2)} d\omega(z) \right)^{(p-2)/p} \\ &< \infty, \end{aligned}$$

by Lemma 5 and the fact $F \in H(p, \Gamma)$ with $p > 2$. Thus the proof will be complete once (4.2) is established. This shall be done in §5.

In case Γ is infinitely generated of convergence type and $F \in H(p, \Gamma)$ ($p \neq 2$), similar methods to those above yield

PROPOSITION 7. *If Γ is of convergence type and $F(z) = \sum_{n=0}^{\infty} a_n z^n \in H(p, \Gamma)$, then*

- (i) $\{a_n n^{-r}\} \in l_2$ for all $r > 1$ if $2 < p < \infty$,
- (ii) $\{a_n n^{-(p+2)/2p}\} \in l_2$ if $1 < p \leq 2$.

The idea of the proof is to show that (1.3) is finite if $s=2$ and $t > 1$ in case (i) and $s=2$ and $t=2/p$ in case (ii). Then using the fact $\beta(p, q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$ and $\Gamma(x)/\Gamma(x+a) \sim x^{-a}$ the result follows by Parseval's formula.

5. An auxiliary result. In the proof of Theorem 6 we merely asserted (4.2). To prove this assertion we need to introduce the following notation analogous to that of [5]. Given $z, \zeta \in U$, we define

$$k(z, \zeta) = \pi^{-1}(1 - z\bar{\zeta})^{-2}$$

and

$$\begin{aligned}\alpha(z, \zeta) &= \theta(k(z, \zeta), \Gamma) \\ &= \sum_{T \in \Gamma} k(Tz, \zeta) T'(z).\end{aligned}$$

It is known (cf. [7, Theorem 1]) that (every arrangement of) the series defining $\alpha(z, \zeta)$ converges, for fixed ζ in U , uniformly on compact subsets of U . Moreover since $k(z, \zeta)$ is a bounded analytic function for each ζ in U it follows that $\alpha(z, \zeta) \in H(p, \Gamma)$ ($1 < p \leq \infty$) whenever Γ is finitely generated. Also we have

PROPOSITION 8. *Suppose Γ is finitely generated of the second kind and $F \in H(p, \Gamma)$ ($1 < p < \infty$). Then*

$$(5.1) \quad \iint_{\Omega} F(z) \overline{\alpha(z, \zeta)} \, dx \, dy = F(\zeta).$$

PROOF. Formally we have

$$\begin{aligned}\iint_{\Omega} F(z) \overline{\alpha(z, \zeta)} \, dx \, dy &= \iint_{\Omega} F(z) \sum_{T \in \Gamma} \overline{k(Tz, \zeta)} \overline{T'(z)} \, dx \, dy \\ &= \iint_U F(z) \overline{k(z, \zeta)} \, dx \, dy = F(\zeta).\end{aligned}$$

To complete the proof it suffices to show that

$$I = \iint_U |F(z)| |k(z, \zeta)| \, dx \, dy < \infty.$$

This follows immediately from the fact that $k(z, \zeta)$ is a bounded analytic function of z for each ζ in U and hence

$$\begin{aligned}I &\leq \|k(z, \zeta)\|_{\infty} \iint_{\Omega} |F(z)| \sum |T'(z)| \, dx \, dy \\ &= \|k(z, \zeta)\|_{\infty} \iint_{\Omega} |F(z) \lambda^{-1}(z)| \left(\sum (1 - |Tz|^2) \right) d\omega(z).\end{aligned}$$

Now Hölder's inequality, (4.1) and the fact that $F \in H(p, \Gamma)$ yields the desired result.

Note 1. If Γ is finitely generated of the second kind then an analogous

argument to that in [6] yields

$$(5.2) \quad 0 \leq \alpha(\zeta, \zeta) = \iint_{\Omega} |\alpha(z, \zeta)|^2 dx dy \leq C(1 - |\zeta|^2)^{-2},$$

$$(5.3) \quad |\alpha(z, \zeta)| \leq C(1 - |z|^2)^{-1}(1 - |\zeta|^2)^{-1}.$$

Note 2. If one developed an L_p theory for automorphic forms of dimension $-2q$, q an arbitrary real ($q > 1$), as in Drasin [2], then the above proposition has an analogous statement (viz. Theorem 3 of [2]).

We can now show

PROPOSITION 9. *If Γ is finitely generated of the second kind then $H(p, \Gamma) \subseteq H(q, \Gamma)$ ($1 < p \leq q \leq \infty$).*

PROOF. We first note that it suffices to show

$$(5.4) \quad H(p, \Gamma) \subseteq H(\infty, \Gamma), \quad 1 < p < \infty,$$

for then, if $q > p$, it follows that

$$\begin{aligned} \iint_{\Omega} |F\lambda^{-1}|^q d\omega(z) &= \iint_{\Omega} |F\lambda^{-1}|^{q-p} |F\lambda^{-1}|^p d\omega(z) \\ &\leq \|F\|_{\infty}^{q-p} \|F\|_p^p < \infty. \end{aligned}$$

To see that (5.4) holds we first assume that $1 < p \leq 2$. Then by (5.1) we have for $F \in H(p, \Gamma)$ and $1/p + 1/p' = 1$

$$\begin{aligned} |F(\zeta)| &= \left| \iint_{\Omega} F(z) \overline{\alpha(z, \zeta)} dx dy \right| \\ &\leq \left(\iint_{\Omega} |F\lambda^{-1}|^p d\omega(z) \right)^{1/p} \left(\iint_{\Omega} |\alpha(z, \zeta)|^{p'} \lambda^{-p'} d\omega(z) \right)^{1/p'} \\ &= \|F\|_p \left(\iint_{\Omega} |\alpha(z, \zeta)|^{p'-2} |\alpha(z, \zeta)|^2 \lambda^{-p'+2} dx dy \right)^{1/p'} \\ &\leq \|F\|_p C^{2-(1/p')}(1 - |\zeta|^2)^{-((p'-2)/p')}(1 - |\zeta|^2)^{2/p'} \\ &= \|F\|_p C^{2-(1/p')}(1 - |\zeta|^2)^{-1} \end{aligned}$$

by Hölder's inequality (5.2) and (5.3). This completes the proof in case $1 < p \leq 2$.

Now we note that an analogous argument based upon a similar reproducing formula for p -integrable holomorphic automorphic forms G of dimension $-2q < -2$ (cf. Drasin [2]) would show that such forms are

bounded if $1 < p \leq 2$, i.e.,

$$\sup_{z \in U} |G(z)| (1 - |z|^2)^q < \infty.$$

Hence, we let $p > 2$ and $F \in H(p, \Gamma)$, and choose an integer m such that $m < p \leq m + 1$. It follows immediately that $F^m(z)$ is a p/m integrable form of dimension $-2m$ and since $1 < p/m \leq 2$ the remark above yields that $|F^m(z)|(1 - |z|^2)^m \leq C$ for all z in U , i.e.

$$\sup_{z \in U} |F(z)| (1 - |z|^2) \leq C^{1/m} < \infty.$$

This completes the proof.

REMARK 1. We note that it is easy to see that if one used an arbitrary system of factors of automorphy $\{\rho(z, T): T \in \Gamma\}$ instead of $T'(z)$ then the statements and results of all of the above theorems go through exactly as is.

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