

ADDITIONS AND CORRECTIONS TO "ON THE IDEAL
STRUCTURE OF THE ALGEBRA
OF RADIAL FUNCTIONS"

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ABSTRACT. The corrections and additions are made in the context of Hankel transforms which generalize the Fourier transforms of radial functions. The following question is studied: given two closed ideals I_1 and I_2 in the algebra of Hankel transforms such that both have the same spectrum and $I_1 \subset I_2$, when is there a closed ideal I such that $I_1 \subset I \subset I_2$?

1. **Introduction.** Professor Aharon Atzmon has been kind enough to point out a counterexample to the theorem in the paper mentioned in the title. That theorem was to have been an analogue for radial functions to the following theorem of Helson [2] which is stated here for the algebra L of integrable functions on R^n (see also [3, p. 183]).

THEOREM. *Suppose $I_1 \subset I_2$ (\subset denotes proper inclusion) where I_1 and I_2 are close ideals of L such that*

$$\{y \mid \hat{f}(y) = 0 \text{ for every } f \in I_1\} = \{y \mid \hat{f}(y) = 0 \text{ for every } f \in I_2\};$$

then there is another closed ideal I of L such that $I_1 \subset I \subset I_2$.

In this paper, we give the counterexample cited by Atzmon and a new theorem. The discussion will be carried out in the language of Hankel transforms; see [8, pp. 236–237] for definitions and notation.

When n is a natural number and $\nu = (n-2)/2$, A_ν is isometrically isomorphic to the algebra L_r of radial integrable functions on R^n with the usual convolution. Indeed, it consists, essentially, of the restrictions of the members of L_r to a ray starting at the origin; and in fact, the Hankel transforms of the functions in A_ν are exactly the restrictions to such a ray of the Fourier transforms of the functions in L_r .

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In the balance of the paper we assume ν is an arbitrary but fixed real number no smaller than $-\frac{1}{2}$. We let k be the greatest integer not exceeding $\nu + \frac{1}{2}$. If I is a closed ideal in A_ν we set $\hat{I} = \{f \mid f \in I\}$ and we say f belongs to \hat{I} locally at y if y has a neighborhood in $[0, \infty)$ on which f agrees with a member of \hat{I} . (The term neighborhood will always refer to a bounded subset of $[0, \infty)$ open in the usual topology.)

The functions in A_ν all have k continuous derivatives on $(0, \infty)$ (see [6] or [9]) and the functionals $f \rightarrow f^{(j)}(y)$ are all continuous for $0 \leq j \leq k$ and $y > 0$. For $f \in A_\nu$ define

$$Z^{(j)}(f) = \{y \mid f^{(j)}(y) = \dots = f^{(k)}(y) = 0\}$$

and if I is an ideal, let

$$Z^{(j)}(I) = \bigcap_{f \in I} Z^{(j)}(f).$$

Since $f^{(j)}$ are continuous for $f \in A_\nu$, all of these sets are closed. We write $Z(I) = Z^{(0)}(I)$ and $Z(f) = Z^{(0)}(f)$, and call $Z(I)$ the spectrum of I . If E is closed, the collection of all closed ideals with spectrum E will be called $\mathcal{I}(E)$. $I_0(E)$ will denote the smallest closed ideal containing all functions f such that E is interior to $Z(f)$. If E is closed and $j = 0, 1, \dots, k$, we define the closed ideals

$$I^{(j)}(E) = \{f \mid f^{(j)}(y) = \dots = f^{(k)}(y) = 0 \text{ all } y \in E\};$$

we write $I(E)$ for $I^{(0)}(E)$; $I(E)$ is the largest ideal in $\mathcal{I}(E)$.

2. The counterexample. Let E consist of a single positive real number y . Since the infinitely differentiable functions with compact support in $[0, \infty)$ are dense in $(A_\nu)^\wedge$, it follows that the ideals

$$I(E) = I^{(0)}(E) \supset I^{(1)}(E) \supset \dots \supset I^{(k)}(E)$$

are all distinct. It follows from [9, Lemma IV] that these are the only ideals of A_ν with spectrum E . Thus at least in the case $\nu \geq \frac{1}{2}$ when E consists of a single positive real number the theorem of [7] is false.

In the balance of the paper, we will investigate the ideal structure of A_ν in an attempt to discover when the structure is discrete; i.e., when there are pairs of distinct closed ideals between which no closed ideals can be found.

3. Additional definitions and notation. If E is a closed subset of $[0, \infty)$ then we say that E has *relative approximate identities* if whenever $f \in A_\nu$ and $E \subseteq Z^{(k)}(f)$ there is a sequence of functions $\{v_n\}$ such that E is interior to $Z(v_n)$ and $\|v_n * f - f\| \rightarrow 0$ as $n \rightarrow \infty$. $\{v_n\}$ is called an *approximate identity for f relative to E* .

If E is any subset of $[0, \infty)$, $\mathcal{C}E$ is its closure, and E' , called the derived set of E , is the set of limit points of E .

We say a partially ordered set P has a discrete structure if given any $x \in P$, there is a countable or finite subset $B(x)$ such that

(a) if $z \in P$ and $z > x$ ($z < x$), then there is $y \in B(x)$ such that $z \geq y > x$ ($z \leq y < x$);

(b) if $y \in B(x)$ and $y > x$ ($y < x$) there is no $z \in P$ such that $y > z > x$ ($y < z < x$).

Two partially ordered sets are isomorphic if there is an order-preserving bijection between them. If E is any subset of $[0, \infty)$, let $\mathcal{W}(E)$ consist of all functions ϕ such that $\phi: E \rightarrow \{0, 1, \dots, k\}$ and $\phi(0) = 0$ if $0 \in E$. We define $I(\phi)$ to be the collection of all $f \in A_v$ such that

$$\hat{f}(x) = \hat{f}^{(0)}(x) = \hat{f}'(x) = \dots = \hat{f}^{(\phi(x))}(x) = 0$$

for every $x \in E$. It is easy to see that $I(\phi) \in \mathcal{S}(\mathcal{C}E)$. If ϕ and ψ are in $\mathcal{W}(E)$, we say that $\phi \geq \psi$ if $\phi(x) \leq \psi(x)$ for every $x \in E$; if this is the case, it follows that $I(\phi) \supseteq I(\psi)$; $\mathcal{W}(E)$ is obviously discrete under this ordering whenever E is countable or finite.

REMARK. The relative approximate identity is generally a weaker notion than that of approximate identity: a closed set E has an approximate identity if there is a sequence of functions $\{v_n\}$ such that E is interior to $Z(v_n)$ for $n = 1, 2, \dots$ and such that, if $f \in I(E)$, then $\|f * v_n - f\| \rightarrow 0$ as $n \rightarrow \infty$ (cf. [3, pp. 48-51]). The existence of an approximate identity implies that $I(E) = I_0(E)$ so that $\mathcal{S}(E)$ consists of exactly one ideal. The question of whether the converse of this is true is still open.

4. Main result. We are now in a position to state a theorem which gives some description of the ideal structure of A_v .

THEOREM. *Let E be a closed subset of $[0, \infty)$, let E' be the derived set of E , and let F be the set of isolated points of E . Suppose E' has relative approximate identities. Then the mapping $\phi \rightarrow K(\phi) = I(\phi) \cap I(E)$ is an isomorphism between $\mathcal{W}(F)$ and $\mathcal{S}(E)$, and so $\mathcal{S}(E)$ is discrete.*

The proof of the theorem will be given in §6. We note that there exists a sequence $\{k_n\}$ of functions such that $\|k_n * f - f\| \rightarrow 0$ for every $f \in A_v$, so the hypotheses above hold when E' is empty and we have the

COROLLARY. *If E is a discrete subset of $[0, \infty)$ then $\mathcal{S}(E)$ is isomorphic to $\mathcal{W}(E)$, and so $\mathcal{S}(E)$ is discrete.*

5. Lemmas. The following lemmas describe the ideal structure of A_v and contain results which will be useful in proving the theorem.

LEMMA 1. Suppose $f(0)=0$; then there is a sequence of functions $\{v_n\}$ such that $\hat{v}_n=0$ on a neighborhood of 0 and such that $\|f * v_n - f\| \rightarrow 0$.

This is Lemma 3.4.17 of [5], which is adapted from [3, pp. 48-51].

LEMMA 2. If $y_0 > 0$, $\mathcal{J}(y_0)$ consists of the $k+1$ distinct ideals $I^{(0)}(y_0), \dots, I^{(k)}(y_0)$; in particular $I_0(y_0) = I^{(k)}(y_0)$.

The lemma is proved easily with the use of Lemma IV of [9] and Lemma 4 of [7].

LEMMA 3. (a) If $f \in A_v$ and \hat{f} does not vanish on some compact subset K of $[0, \infty)$, then there is $g \in A_v$ such that $\hat{g}\hat{f}=1$ on K .

(b) If f and g are in A_v and $Z(g)$ is interior to $Z(f)$, then, if \hat{f} has compact support, $f=g * h$ for some $h \in A_v$.

The lemma is easily proved using the techniques of [1, pp. 119-130].

LEMMA 4. If I is a closed ideal of A_v and $f \in A_v$ and if \hat{f} is in \hat{I} locally at each point of $[0, \infty)$, then $f \in I$.

This can be proved by using the technique of [3, Lemma 6.2.6].

LEMMA 5. If $f \in A_v$ and I is a closed ideal of A_v , then $\hat{f} \in \hat{I}$ locally at y_0 if either of the following two conditions hold:

- (i) $y_0 \notin Z(I)$.
- (ii) y_0 is an interior point of $Z(f)$.

PROOF. (a) If $y_0 \notin Z(I)$, then there is $g \in I$ such that $\hat{g}(y_0) \neq 0$. The continuity of g and Lemma 3 ensure the existence of a function $h \in A_v$ such that $\hat{h}\hat{g}=1$ on a neighborhood U of y_0 . Then \hat{f} coincides with $\hat{f}\hat{h}\hat{g}$ in U and the latter function is the transform of $(f * h) * g$ which is in I , so \hat{f} is in \hat{I} locally at y_0 .

(b) follows since I contains the constant zero function.

LEMMA 6. If I is a closed ideal of A_v , $f \in A_v$ and $Z(I) \subseteq Z^{(k)}(f)$, then the set

$$C = \{y \mid \hat{f} \text{ is not in } \hat{I} \text{ locally at } y\}$$

is closed, and if the derived set of $Z(f)$ has relative approximate identities, then C has no isolated points; thus C is either empty or perfect.

PROOF. The complement of C is open, so C is closed. Suppose by way of contradiction that y_0 is an isolated point of C . Let W be a neighborhood of y_0 which contains no other point of C and let $h \in A_v$ be such that $\hat{h}=1$ on a neighborhood of y_0 and $\hat{h}=0$ off of a compact subset contained in W . Since $y_0 \in C$, then $y_0 \in Z(I)$ by Lemma 5, so $y_0 \in Z^{(k)}(f)$. We treat two cases depending on whether or not y_0 is an isolated point of $Z(f)$.

Suppose y_0 is a limit point of $Z(f)$; a repeated application of Rolle's theorem shows that the derived set of $Z(f)$ is contained in $Z^{(k)}(f)$. Let $\{v_n\}$ be an approximate identity for f relative to the derived set of $Z(f)$; then v_n vanishes in a neighborhood of y_0 . We claim $f_n = v_n * h * f$ belongs to I locally at every point of $[0, \infty)$. To see this, note y_0 is interior to $Z(v_n)$ and if $y \notin W$, y is interior to $Z(h)$ so if $y \notin W$ or $y = y_0$, y is interior to $Z(f_n)$, and, by Lemma 5, f_n belongs to I locally at y . By assumption, f belongs to I locally at every point of $W - \{y_0\}$, hence so does f_n . Thus f_n belongs to I locally at every point of $[0, \infty)$, so $f_n \in I$ by Lemma 4. Since $v_n * f \rightarrow f$, $f_n \rightarrow h * f$, and since I is closed, $h * f \in I$. But $h * f = f$ on a neighborhood of y_0 , so f belongs to I locally at y_0 , whence $y_0 \notin C$ which is a contradiction.

Suppose now that y_0 is an isolated point of $Z(f)$. If $y_0 = 0$ we use Lemma 1 and proceed as above. If $y_0 > 0$, let W and h be chosen as above to satisfy the additional condition that $\mathcal{C}W \cap Z(f) = \{y_0\}$. It is an easy matter to construct a function g such that $\hat{g} = f$ on W and such that $Z(g) = \{y_0\}$. Then the closed ideal generated by g must be $I^{(k)}(y_0)$ by Lemma 2. Now, also by Lemma 2, $I^{(k)}(\{y_0\}) = I_0(\{y_0\})$ so we can find g_n such that y_0 is interior to $Z(g_n)$, \hat{g}_n has compact support, and $\|g_n - g\| \rightarrow 0$. It follows from Lemma 3 that there are functions v_n such that $g_n = v_n * g$. In particular, each \hat{v}_n vanishes on a neighborhood of y_0 , and on W , $\hat{v}_n f = \hat{v}_n \hat{g}$. We can now show, as in the first part of the proof, that $f_n = \hat{h} \hat{v}_n f$ belongs to I locally at every point y of $[0, \infty)$, so $f_n \in I$ and finally $f \hat{h} \hat{v}_n = \hat{g} \hat{h} \hat{v}_n$ and $\|g * v_n \rightarrow g\|$ so $g * h \in I$ but $\hat{g} \hat{h} = f$ on a neighborhood of y_0 ; thus f is in I locally at y_0 which is a contradiction.

6. Proof of theorem. We first show that the mapping is injective. Suppose ϕ and ψ are in $\mathcal{W}(F)$. If $\phi \neq \psi$ then $\phi(x) \neq \psi(x)$ for some $x \in F$; $x \neq 0$, since if $0 \in F$, $\phi(0) = 0 = \psi(0)$. Let W be a neighborhood of x such that $W \cap E = \{x\}$. Assume $\phi(x) < \psi(x)$ and let g be a function with compact support and infinitely many derivatives such that

$$g = 0 \quad \text{off } W,$$

$$g(x) = g'(x) = \cdots = g^{(\phi(x))}(x) = 0 \quad \text{and} \quad g^{(\psi(x))}(x) \neq 0.$$

Then $f = \hat{g} \in A_v$, as can be seen by repeated integration by parts and, in fact $\hat{f} = g$ and $f \in K(\phi)$, but $f \notin K(\psi)$, so $K(\phi) \neq K(\psi)$.

Now suppose $I \in \mathcal{S}(E)$; we define ϕ by setting $\phi(0) = 0$ if $0 \in F$, and $\phi(y) = \max\{j | y \in Z^{(j)}(I)\}$ if $y \in F$ and $y \neq 0$. Then $I \subseteq K(\phi)$. Suppose $f \in K(\phi)$, then $E' \subseteq Z^{(k)}(f)$; let $\{v_n\}$ be an approximate identity for f relative to E' .

We claim $f \hat{v}_n$ is contained in I locally at every point of $[0, \infty)$. There are three cases: (i) $y \notin E$, (ii) $y \in F$, (iii) $y \in E'$. Case (i) is disposed of by

Lemma 5. To handle case (ii) let W be a neighborhood of y such that $W \cap E = \{y\}$, let $m = \max\{j | f \in I^{(j)}(\{y\})\}$; then there is $g \in I$ such that $Z(g) \cap W = \{y\}$, $\hat{g}(y) = \cdots = \hat{g}^{(m)}(y) = 0$, and, unless $m = k$, $\hat{g}^{(m+1)}(y) \neq 0$; then $(I^{(m)}(\{y\}))^\wedge$ coincides on W with the closed ideal generated by g because of Lemma 2. But $f * v_n \in I^{(m)}(\{y\})$ so $f \hat{v}_n$ coincides with a function from \hat{I} on W . Finally, if $y \in E'$, y is interior to $Z(v_n)$ so $f * v_n$ belongs to I locally at y . We have proved our claim so $f * v_n \in I$ by Lemma 4, but $\|f * v_n - f\| \rightarrow 0$ and I is closed, so $f \in I$, therefore $I = K(\phi)$.

That the mapping is order preserving is obvious, so the proof is complete.

7. Additional remarks and open questions. We recall that when n is a natural number and $\nu = \frac{1}{2}(n-2)$ the algebras L_ν and A_ν are isomorphic so our results give information about the ideal structure of L_ν . Moreover to each ideal L_ν corresponds a unique rotation invariant ideal of L (see [7, Lemma 4]), so we also get information about that class of ideals.

We conclude with some remarks and questions about the ideal structure of A_ν and relative approximate identities.

1. The hypothesis in the theorem that E have approximate relative identities was chosen to assure that $I_0(E) = I^{(k)}(E)$. Is the converse true? In particular, if E consists of a single point $y_0 > 0$, does E have a relative approximate identity? (With appropriate change in the definition, the question can be asked when $y_0 = 0$. The answer to this is "yes" and can be found in [5, Theorem 3.3.2].) Might it be the case that there is a fixed sequence $\{v_n\}$ of functions in A_ν such that y_0 is interior to $Z(v_n)$ and $\|v_n * f - f\| \rightarrow 0$ as $n \rightarrow \infty$ for every $f \in I^{(k)}(\{y_0\})$?

2. If $I_0(E) \neq I^{(k)}(E)$ must there be an ideal between them? Can the ideal structure be dense in the sense that, if $I_1 \subset I_2$ are distinct ideals in $\mathcal{I}(E)$, then there is an ideal J such that $I_1 \subset J \subset I_2$, at least if $I_2 \subseteq I^{(k)}(E)$?

3. We observe that, if E is closed, then $E' \subseteq Z^{(k)}(I)$ for $I \in \mathcal{I}(E)$. What can be concluded about the structure of $\mathcal{I}(E)$ given the structure of $\mathcal{I}(E')$? If F is the set of isolated points of E then does the structure of $\mathcal{I}(E)$ above $I^{(k)}(E)$ become complicated if that between $I_0(E)$ and $I^{(k)}(E)$ does? To be exact, we ask if the following sort of theorem might hold: "If $I \in \mathcal{I}(E)$ there is a unique $J \in \mathcal{I}(E')$ and $\phi \in W(F)$ such that $I = J \cap I(\phi)$."

4. If $I^{(k)}(E) = I(E)$ does it follow that $I^{(k)}(E) = I_0(E)$? The converse is, of course, false by Lemma 2.

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