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ON BOREL MEASURES AND BAIRE'S CLASS 3

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ABSTRACT. Let S be a complete and separable metric space and μ a σ-finite, complete Borel measure on S. Let Φ be the family of all real-valued functions, continuous μ-a.e. Let $B_\alpha(\Phi)$ be the functions of Baire's class α generated by Φ. It is shown that if μ is not a purely atomic measure whose set of atoms form a dispersed subset of S, then $B_2(\Phi)\neq B_{\omega_1}(\Phi)$, where ω_1 denotes the first uncountable ordinal.

If Φ is a family of real-valued functions defined on a set S, then, $B(\Phi)$, the Baire system of functions generated by Φ is the smallest subfamily of R^X which contains Φ and which is closed under the process of taking pointwise limits of sequences. The family $B(\Phi)$ can be generated from Φ as follows: Let $B_0(\Phi)=\Phi$ and for each ordinal α, let $B_\alpha(\Phi)$ be the family of all pointwise limits of sequences taken from $\bigcup_{\gamma<\alpha} B_\gamma(\Phi)$. Thus, $B_{\omega_1}(\Phi)$ is the Baire system of functions generated by Φ, where ω_1 is the first uncountable ordinal. The Baire order of a family Φ is the first ordinal α such that $B_\alpha(\Phi)=B_{\alpha+1}(\Phi)$.

Kuratowski has proved that if S is a metric space and Φ is the family of all real-valued functions on S which are continuous except for a first category set, then the order of Φ is 1 and $B_1(\Phi)$ is the family of all functions which have the Baire property in the wide sense [**1**, p. 323].

Let S be a complete separable metric space, let μ be a σ-finite, complete Borel measure on S and let Φ be the family of all real-valued functions on S, whose set of points of discontinuity is of μ-measure 0. In [**3**], it was shown that the order of Φ is 1 if and only if μ is purely atomic and the set of atoms of μ is a dispersed [**7**] (scattered) subset of S. Thus, as far as the Baire order problem is concerned the notion of first category and measure 0 cannot, in general, be interchanged.

The purpose of this paper is to prove that if μ is not a purely atomic measure whose atoms form a dispersed set, then the Baire order of Φ is at least 3. This will be accomplished by exhibiting a function in $B_3(C(S))$ which is not in $B_2(\Phi)$. Of course, $B_3(C(S))$ is a subfamily of $B_3(\Phi)$.

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In what follows suppose μ is not a purely atomic measure whose set of atoms form a dispersed set. In order to prove the main theorem the following technical lemma is employed.

LEMMA. *There is a perfect set M of finite measure such that if V is an open set intersecting M, then $\mu(M \cap V) > 0$ and there is a set of sequences, $\{T_{np}\}_{p=1}^{\infty}$, $n=1, 2, 3, \cdots$, such that*

(1) *$\{T_{1p}\}_{p=1}^{\infty}$ is a sequence of disjoint, perfect, nowhere dense subsets of M such that $T_1 = \bigcup_{p=1}^{\infty} T_{1p}$ is a dense subset of M and for each p, if V is an open set intersecting T_{1p}, then $\mu(T_{1p} \cap V) > 0$;*

(2) *for each n, $\{T_{n+1,p}\}_{p=1}^{\infty}$ is a sequence of disjoint perfect subsets of M such that for each k, $T_{n+1,k}$ is a subset of some term of $\{T_{np}\}_{p=1}^{\infty}$ and is nowhere dense with respect to that set, and if V is an open set intersecting $T_{n+1,k}$ then $\mu(T_{n+1,k} \cap V) > 0$; and*

(3) *for each n and p, the union of all the sets in $\{T_{n+1,k}\}_{k=1}^{\infty}$ which are subsets of T_{np} is dense in T_{np}.*

PROOF. Since the support of μ is a closed set, it can be decomposed into a perfect part, P, and a countable scattered (dispersed) part D. From the assumptions made concerning μ, we have that the perfect part of this decomposition is nonempty.

It follows that there is a perfect subset M of P of finite measure such that if V is an open set which meets M, then $\mu(M \cap V) > 0$.

If H is a family of disjoint, perfect, nowhere dense subsets of M, each having positive μ-measure then the family H is countable. Let G be the collection of all such families H which have the additional property that if an open set V meets some member K of H, then $\mu(V \cap K) > 0$. Let the collection G be partially ordered by inclusion. Since every totally ordered subsystem of G has an upper bound, G has a maximal element: $\{T_{1p}\}_{p=1}^{\infty}$.

Now, suppose that $T_1 = \bigcup_{p=1}^{\infty} T_{1p}$ is not dense in M. Let U be an open set which meets M such that Cl U does not intersect T_1. The set $\mathrm{Cl}(U \cap M)$ is a perfect subset of M and $\mu(\mathrm{Cl}(U \cap M)) > 0$. It follows that there is a perfect nowhere subset K of M, lying in $\mathrm{Cl}(U \cap M)$ such that if V is an open set which meets K, then $\mu(V \cap K) > 0$. This contradicts the maximality of the family $\{T_{1p}\}_{p=1}^{\infty}$. Thus, T_1 is dense in M.

Arguments, similar to the preceding one, can be given to complete the proof of the lemma.

THEOREM. *Let S be a complete separable metric space, let μ be a σ-finite, complete Borel measure on S and let Φ be the family of all real-valued functions on S which are continuous μ-a.e. If the Baire order of Φ is not* 1, *then the order of Φ is at least* 3.

PROOF. Let M be a perfect subset of S and $\{T_{np}\}_{p=1}^{\infty}$, $n=1, 2, 3, \cdots$, a set of sequences of subsets of M satisfying the conditions of the conclusion of the lemma. Let $T=\bigcap_{n=1}^{\infty} T_n$. The set T is an $F_{\sigma\delta}$ set and $\mathscr{E}_T$, the characteristic function of T is in Baire's class 3 ([**8**], [**9**]).

Suppose $\mathscr{E}_T$ is in $B_2(\Phi)$. By Theorem 3 of [**4**], there is an F_σ set K such that $\mu(K)=0$ and a function g in Baire's class 2 such that if x is in $S-K$, $g(x)=\mathscr{E}_T(x)$. Hence, $\mathscr{E}_T|K'$ is in Baire's class 2 with respect to K', the complement of K. So, $T\cap K'$ is a $G_{\delta\sigma}$ set with respect to K'. There is a $G_{\delta\sigma}$ subset A of S such that $A\cap K'=T\cap K'$. Since K' is a G_δ set in S, $T\cap K'$ is a $G_{\delta\sigma}$ set in S.

We have $T'=T_1'\cup(T_1-T_2)\cup(T_2-T_3)\cup\cdots$. Since T_n-T_{n+1} is a $G_{\delta\sigma}$ set, for each n and T_1' and K are $G_{\delta\sigma}$ sets, it follows that $T'\cup K$, the complement of $T\cap K'$, is a $G_{\delta\sigma}$ set. Thus, $T'\cup K$ is an ambiguous set of class 2.

It follows that there is a sequence $\{A_n\}_{n=1}^{\infty}$ of ambiguous sets of class 1 such that

$$T'\cup K=\bigcup_{n=1}^{\infty}(A_n\cap A_{n+1}\cap\cdots)=\bigcap_{n=1}^{\infty}(A_n\cup A_{n+1}\cup\cdots) \qquad [\mathbf{6}, \text{p. } 355].$$

Let $\{A_{n_1 i}\}_{i=1}^{\infty}$ be the subsequence of $\{A_n\}_{n=1}^{\infty}$ consisting of all terms which intersect T' and for each $k>1$, let $\{A_{n_k i}\}_{i=1}^{\infty}$ be the subsequence of $\{A_n\}_{n=1}^{\infty}$ consisting of all terms which intersect $T_{k-1}-T_k$ and having subscript $\geq k$.

For each k, let $B_k=\bigcup_{i=1}^{\infty} A_{n_k i}$. It follows that for each k, B_k is an F_σ set, B_1 contains T_1' and if $k>1$, B_k-contains $T_{k-1}-T_k$. Also, it follows that $\limsup_{k\to\infty} B_k$ is a subset of $\limsup_{n\to\infty} A_n$, which is $T'\cup K$.

Let $K=\bigcup_{n=1}^{\infty} F_n$, where for each n, F_n is a closed set of measure 0 and F_{n+1} contains F_n.

Since B_1 is an F_σ set containing T_1', and T_1 is of the first category with respect to P, there is an open set C_1 intersecting P such that $\mathrm{Cl}(C_1\cap P)$ is a subset of B_1. Since F_1 is closed and $\mu(F_1)=0$, $F_1\cap P$ is a closed, nowhere dense subset of P. Let S_1 be a spherical ball of radius less than 1, intersecting P such that $\mathrm{Cl}(S_1\cap P)$ is a subset of $C_1\cap P$ and $\mathrm{Cl}(S_1\cap P)$ does not intersect F_1.

Since T_1 is a dense subset of P, there is a positive integer n_1 such that T_{1n_1}, intersects S_1. Thus, $S_1\cap T_{1n_1}$, is a dense in itself subset of T_{1n_1}, and $H_1=\mathrm{Cl}(S_1\cap T_{1n_1})$, is a perfect subset of T_{1n_1}, such that if $\mathscr{O}$ is an open set intersecting H_1, then $\mu(\mathscr{O}\cap H_1)>0$.

As B_2 is an F_σ set containing T_1-T_2 and T_2 is of the first category with respect to H_1, B_2 is not of the first category with respect to H_1. There is an open set C_2 lying in S_1 and intersecting H_1 such that $\mathrm{Cl}(H_1\cap C_2)$ is a subset of B_2. Since F_2 is closed and $\mu(F_2)=0$, $F_2\cap H_1$ is a closed, nowhere dense subset of H_1. Let S_2 be a spherical ball of radius less than $\frac{1}{2}$

intersecting H_1 such that $\mathrm{Cl}(S_2)$ subset of C_2 and $\mathrm{Cl}(S_2 \cap H_1)$ does not intersect F_2.

As $T_2 \cap T_{1n_1}$ is a dense subset of T_{1n_1}, there is a positive integer n_2 such that T_{2n_2} intersects $S_2 \cap H_1$. Then T_{2n_2} is a subset of T_{1n_1}, $S_2 \cap T_{2n_2}$ is a dense in itself subset of $S_2 \cap H_1$ and $H_2 = \mathrm{Cl}(S_2 \cap T_{2n_2})$ is a perfect subset of T_{2n_2} such that if O is an open set intersecting it, then $\mu(O \cap H_2) > 0$.

Suppose $k > 1$ and sets C_i, S_i, and T_{in_i}, $1 \leqq i \leqq k$, have been defined having the following properties:

(1) C_k is an open set lying in S_{k-1} such that $\mathrm{Cl}(H_{k-1} \cap C_k)$, where $H_{k-1} = \mathrm{Cl}(S_{k-1} \cap T_{k-1n_{k-1}})$, is a subset of B_k;

(2) S_k is a spherical ball of radius less than $1/k$ intersecting H_{k-1} such that $\mathrm{Cl}(S_k)$ is a subset of C_k and $\mathrm{Cl}(S_k \cap H_{k-1})$ does not intersect F_k;

(3) n_k is a positive integer such that T_{kn_k} intersects $S_k \cap H_{k-1}$.

Now, an argument analogous to the one given above to obtain the sets C_2, S_2, and T_{2n_2} may be used to obtain sets C_{k+1}, S_{k+1}, and $T_{k+1_{n+1}}$ having properties (1), (2), and (3) listed above where $k+1$ is substituted for k.

The sequence $\{H_p\}_{p=1}^{\infty}$ is a monotonically decreasing sequence of closed point sets in the complete and separable space S and the diameter of H_{p+1} is less than $2/(p+1)$. There is a point w common to all the terms of the sequence $\{H_p\}_{p=1}^{\infty}$. As for each p, H_p is a subset of B_p and of T_p and H_p does not intersect F_p, w is in $\limsup B_n$ and w is in $T \cap K'$. But, $\limsup B_n$ is a subset of $T' \cup K$. This is a contradiction. This completes the argument for the theorem.

L. Kantorovitch has shown that in the special case $S = [0, 1]$ and μ is Lebesgue measure, there is a function in Baire's class 2, not in $B_1(\mathscr{F})$ [**10**]. The theorem proved in this paper shows that there is a function in Baire's class 3, not in $B_2(\mathscr{F})$, if μ is not a purely atomic measure having a dispersed set of atoms. It is not difficult to show from results in [**4**] that $B_{\alpha+1}(\mathscr{F}) \neq B_\alpha(\mathscr{F})$ if and only if there is a function f in Baire's class $\alpha+1$ such that if g is in Baire's class α, then the set $(f \neq g)$ is not a subset of an F_σ set of measure 0.

CONJECTURE. If the Baire order of Φ, the family of all real-valued functions continuous a.e. is not 0 or 1, then it is ω_1.

REMARK. To settle this question, calls for some delicate analysis as it is well known that every measurable function f agrees with a function g in Baire's class 2 almost everywhere; however, as has been shown here the topological nature of the set $(f \neq g)$ is very important in this process.

QUESTION. Is there any family Φ whose Baire order is not 0, 1, 2 or ω_1?

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