PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 39, Number 2, July 1973

EFFECTIVE MATCHMAKING AND k-CHROMATIC GRAPHS

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ABSTRACT. In an earlier paper we showed that there is a recursive society, in which each person knows exactly two other people, whose marriage problem is solvable but not recursively solvable. We generalize this result, using a different construction, to the case where each person knows exactly k other people. From this we deduce that for each $k \geqq 2$ there is a recursive $2(k-1)$-regular graph, whose chromatic number is k but which is not recursively k-chromatic.

1. **Graphs, societies, and algorithms.** Following Berge [**1**] a set S of unordered pairs of distinct elements of a set P determines a *graph* $\Gamma=(P, S)$. The elements of P are called *points* or *vertices* of Γ; the elements of S are called *arcs* of Γ. It is not assumed that P or S is finite. Points x and y are said to be *adjacent* if $\{x, y\}$ is an arc. Γ is *k-chromatic* if the points of Γ can be painted with k colors in such a way that no two adjacent points are of the same color. The *chromatic number of* Γ is the smallest number k such that Γ is k-chromatic. Γ is *k-regular* if every point of Γ is adjacent to exactly k points. Γ is called *simple* (or *bipartite*) if there exist disjoint sets B and G such that $P=B\cup G$ and if wherever $\{x, y\} \in S$ then $x \in B$ and $y \in G$ *or* $y \in B$ and $x \in G$. Two distinct arcs are said to be *adjacent* if they have a point in common. A *matching* of a simple graph (B, G, S) is a set W of arcs no two of which are adjacent. Let W be a matching, $B_W=\{b \in B \mid \text{for some } g, \{b, g\} \in W\}$, and $G_W=\{g \in G \mid \text{for some } b, \{b, g\} \in W\}$; W is then said to be a *matching of B_W onto G_W* or a *matching of B_W into G*.

We now recall the more colorful, anthropomorphic terminology of Halmos and Vaughan [**3**]. Let $\Sigma=(B, G, S)$ be a simple graph. We call

Presented in part to the Society, January 24, 1971 under the title *Effective matchmaking (Recursion theoretic aspects of a theorem of Philip Hall)*; received by the editors August 23, 1972.

AMS (MOS) subject classifications (1970). Primary 02F50, 05C15, 02E10; Secondary 05A05.

Key words and phrases. Bipartite graph, k-regular graph, chromatic number, matching, marriage problem, algorithm, recursive function.

[1] Partially supported by NSF Grant GP-23771.

[2] Partially supported by NSF Grant GP-20134.

Σ a *society*, B the *set of boys*, and G the *set of girls*. If b and g are adjacent in the graph Σ we say that b and g are *acquainted* in the society Σ. We call the society Σ a *k-society* if as a graph it is k-regular, so that in a k-society each person knows exactly k people of the opposite sex. The society Σ is said to have a *solvable marriage problem* if there is a matching of B into G, for we can think of the matching as providing, in a monogamous way, a mate for each boy from among the girls he knows. Similarly, Σ is said to have a *symmetric solution to its marriage problem* if there is a matching of B *onto* G

We also associate with the society Σ another graph Γ_Σ as follows. The points of Γ_Σ are the arcs of Σ and the arcs of Γ_Σ are the unordered pairs of adjacent arcs of Σ. If Σ is a k-society then Γ_Σ is a $2(k-1)$-regular graph.

We use three combinatorial lemmas which we state here without proof.

LEMMA 1. *If Σ is a k-society, then there is a symmetric solution to its marriage problem.*

LEMMA 2. *If Σ is a k-society, the chromatic number of Γ_Σ is k.*

LEMMA 3. *Let Σ be a k-society. The set of points of Γ_Σ possessing a common color in a k-coloring of Γ_Σ is a matching of B onto G in Σ. Thus such a set of points is a solution to the symmetric marriage problem of Σ.*

In the finite case, Lemmas 1 and 2 are just restatements of results due to König and P. Hall appearing in Berge [**1**, pp. 92–95]. To prove these lemmas in the infinite case, one can, for example, use O. Ore's extension of the Schroeder-Bernstein theorem (Theorem 1.3.4 in Mirsky [**4**]). Lemma 3 is easily verified directly.

Following Rogers [**5**] a function which is computable by an algorithm or an effective procedure is called a *partial recursive function*. The domain of a partial recursive function is assumed to be a subset of N^m for a fixed m (N is the set of natural numbers) and its range is assumed to be a subset of N. If its domain happens to be all of N^m the partial recursive function θ is called a (*general*) *recursive function*. If x is in the domain of θ we say that $\theta(x)$ is defined; otherwise we say that $\theta(x)$ is undefined. A set is said to be *recursive* if its characteristic function is a recursive function.

The collection of all finite sets of instructions, or algorithms, formulated in a fixed language can be recursively (i.e. effectively) enumerated. Assuming this to be done, ϕ_e denotes the partial recursive function defined by the eth finite set of instructions. Given an argument x and a number e of a set of instructions, it is not possible to determine effectively whether or not $\phi_e(x)$ is defined. However, it is possible, for each n, to determine effectively—in e, x, and n—whether or not $\phi_e(x)$ is defined in n steps, by simply carrying out n steps of the eth algorithm applied to x and observing the outcome. "$\phi_e^n(x)$ is defined" will mean that $\phi_e(x)$ is defined in n steps; in that case the value of $\phi_e^n(x)$ will be $\phi_e(x)$. Furthermore if $\phi_e(x)$ is defined there must be an n such that $\phi_e^n(x)$ is defined—and for all $n' \geqq n$, $\phi_e^{n'}(x)$ is defined and equals $\phi_e(x)$. The formal statements and verifications of these remarks can be found, for example, in Rogers [5, Theorems 1-IV and 1-IX].

In what follows a society will also satisfy the conditions (i) each person knows only finitely many other people (i.e. Σ is locally finite) and (ii) everyone knows someone. If all but (ii) are satisfied by Σ, then Σ will be called a *partial society*. The connected components of a partial society Σ are called the *communities of* Σ.

We say that the society Σ is *recursive* if B is the set of even natural numbers, G is the set of odd natural numbers, and S is a recursive set of ordered pairs. We will use $B(n)$ for $2n$ and $G(n)$ for $2n+1$ and say that $B(n)$ is the nth boy and that $G(n)$ is the nth girl. A recursive society is said to be *recursively* (*symmetrically*) *solvable* if there is a 1-1 (onto) recursive function f such that, for each n, $B(n)$ knows $G(f(n))$.

We say that the graph $\Gamma=(P, S)$ is *recursive* if $P=N$ and S is a recursive set of ordered pairs. Γ is said to be *recursively k-chromatic* if there is a recursive function f of one variable whose range is a subset of $\{0, 1, \cdots, k-1\}$ such that if x is adjacent to y then $f(x)\neq f(y)$.

Let Σ be a recursive society and let j be a 1-1 function which maps S recursively onto N. Define $j(\Gamma_\Sigma)$ to be the graph whose points are N and whose arcs are the pairs $\{j(b, g), j(b', g')\}$ such that $\{(b, g), (b', g')\}$ is an arc of Γ_Σ. Observe that if Σ is a recursive society, $j(\Gamma_\Sigma)$ is a recursive graph. Since $j(\Gamma_\Sigma)$ is isomorphic to Γ_Σ, we know that if Σ is a k-society, $j(\Gamma_\Sigma)$ is a $2(k-1)$-regular graph, and that, by Lemma 2, $j(\Gamma_\Sigma)$ has chromatic number k. Lemma 3 shows that if $j(\Gamma_\Sigma)$ is recursively k-chromatic, then Σ is recursively solvable. These observations show that the following corollary is a consequence of the existence of a recursive k-society which is not recursively solvable. This will be proved in §2.

COROLLARY. *There is a recursive $2(k-1)$-regular graph whose chromatic number is k, but which is not recursively k-chromatic.*

2. **Recursive k-societies without recursive solutions.**

THEOREM. *For each $k\geqq 2$ there is a recursive k-society which is symmetrically solvable but is not recursively solvable.*

PROOF. In the proof we construct a recursive society Σ by stages; at stage n, for each $n>0$, a partial society $\Sigma_n=(B, G, S_n)$, with S_n finite, is effectively defined so that, for each n, $S_n\subseteq S_{n+1}$ and so that $\Sigma=(B, G, \bigcup_{n>0} S_n)$ has the desired properties. Instead of saying "put (x, y)

into S_n" we will say "introduce x to y" or "introduce y to x" at stage n. A person is called a "stranger" at a given point in the construction if he has not yet been introduced to anyone. At the beginning of each stage n of the construction there are numbers a and b (with $a \geqq n$ and $b \geqq n$) such that the first a boys and b girls are not strangers at that point, but the remaining boys and girls are; we will reserve the numbers a and b for this purpose, so that $B(a)$ and $G(b)$ always are the first male and female strangers at the beginning of the appropriate stage of the construction.

For each n, each introduction made during stage n involves at least one person who was a stranger at the beginning of stage n. This feature, together with the effectiveness of the construction of S_n, implies that Σ is recursive. To see this we show how to decide whether or not $B(p)$ knows $G(q)$. Let $n>p$ and $n>q$. Since the first male and female strangers at stage n are $B(a)$ and $G(b)$ where $a \geqq n$ and $b \geqq n$ it follows that $B(p)$ and $G(q)$ have acquaintances in Σ_n. Hence $B(p)$ knows $G(q)$ in Σ if and only if he already knows her in Σ_n. But whether or not he knows her in Σ_n can be effectively determined by effectively reconstructing S_n.

The community of the partial society Σ_{n-1} to which $B(i)$ belongs at the beginning of stage n will be denoted $C_n(i)$. $C_n(i)$ is called *stable* if $C_m(i)=C_n(i)$ for all $m \geqq n$. The remarks in the preceding paragraph show that if $C_n(i)$ is stable, then no member of $C_n(i)$ will ever meet someone new. In particular, if $C_n(i)$ is stable and $B(p)$ and $G(q)$ are in $C_n(i)$ but cannot marry in $C_n(i)$ (i.e. there is no solution to the marriage problem of $C_n(i)$ in which $B(p)$ marries $G(q)$), then $B(p)$ cannot marry $G(q)$ in Σ.

We now define simultaneously the recursive society Σ and k 1-1 recursive functions $r_0, r_1, \cdots, r_{k-1}$ (with pairwise disjoint ranges); at the end of stage n, $r_t(i)$ will be defined for all $i<n$ and $t<k$.

Intuitively, the construction will guarantee that if $\phi_e(r_t(e))$ is defined for all $t<k$ than no solution to the marriage problem of Σ simultaneously marries each $B(r_t(e))$ to the corresponding $G(\phi_e(r_t(e)))$, so that ϕ_e cannot be a solution to the marriage problem of Σ. Since every recursive function is ϕ_e for some e, this implies that the marriage problem of Σ has no recursive solution.

We assume as part of the induction hypothesis that at stage n for each $i<n$ either all $B(r_t(i))$ are in the same community or they are in k different communities. In the former case, the community is a stable one in which each person knows exactly k others. In the latter case there is a p such that for each $t<k$ the community $C_n(B(r_t(i)))$ contains exactly $1+(k-1)k+(k-1)^3k+\cdots+(k-1)^{2p+1}k$ boys and $k+(k-1)^2k+\cdots+(k-1)^{2p}k$ girls, and can be put into 1-1 correspondence with the nodes of the graph g_p below in such a way that boys correspond to nodes marked ♂, girls correspond to nodes marked ♀, $B(r_t(i))$ corresponds to the bottom node,

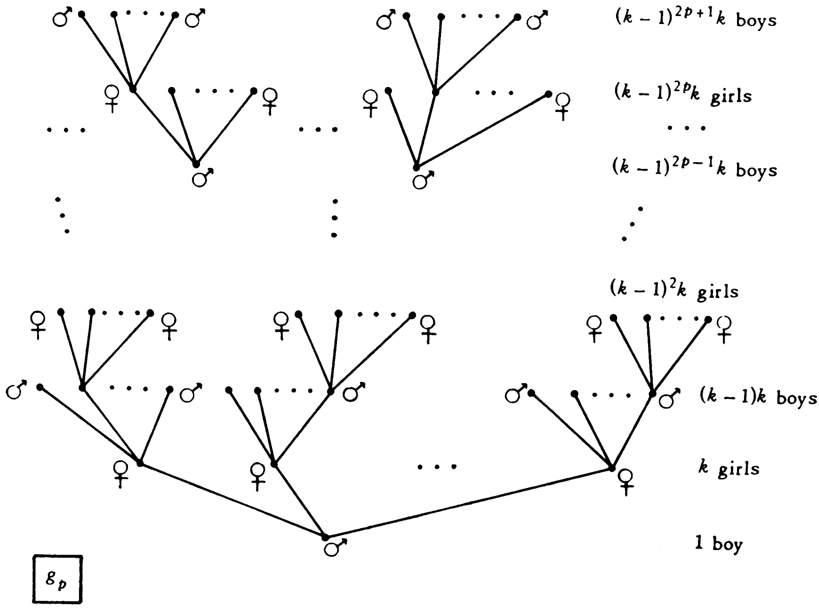


FIGURE 1

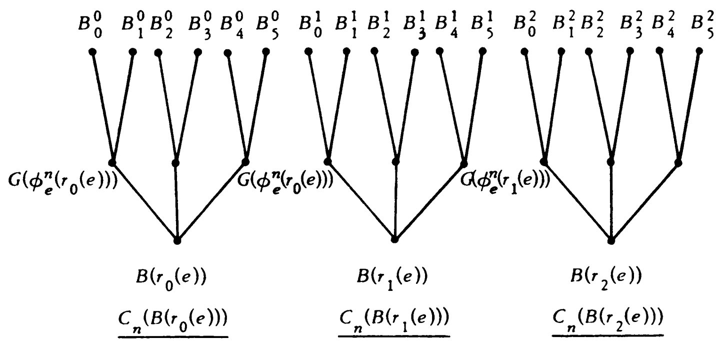
and two nodes are adjacent if and only if the people mapped to these nodes know each other. If this is the case we shall say that $C_n(B(r_t(i)))$ has form g_p. We assume that the definition of the jth row of g_p, for $0 \leqq j \leqq 2p+2$, and of the ith position (from the left) on the jth row of g_p, for $0 \leqq i < (k-1)^{j-1}k$ where $j \geqq 1$, need not be made explicit. It is also clear what we mean when we say that (under a particular correspondence) a certain person of the community C (which has form g_p) is in the ith position of the jth row of C. Note that in a community of form g_p each person except those on the top, i.e. $(2p+2)$th, row know exactly k other people.

Stage $n>0$. Define $r_t(n-1)=a+t$ for each $t<k$ (the first k unused boys) and establish for each $B(r_t(n-1))$ a community containing k additional new girls, and $k(k-1)$ additional new boys, so that it has the form g_0.

Let $n=2^e q$ where q is odd, say $q=2m+1$. If all $B(r_t(e))$ are already in the same community proceed to stage $n+1$. If they are still in different communities, and if either some $\phi_e^n(r_t(e))$ is undefined or all are defined but some $B(r_t(e))$ does not know $G(\phi_e^n(r_t(e)))$, then, since each of the k communities is of the form g_p where $p=m-1$, we introduce each of the $(k-1)^{2p+1}k$ boys in the top row of each community to $k-1$ new girls and each of these $(k-1)\cdot(k-1)^{2p+1}k$ girls to $(k-1)$ new boys, so that the resulting k communities are all of the form g_{p+1}. Finally, we consider the case where the $B(r_t(e))$ are in different communities, where all $\phi_e^n(r_t(e))$ are defined, and where $B(r_t(e))$ knows $G(\phi_e^n(r_t(e)))$ for each $t<k$. We assume that each $C_n(B(r_t(e)))$ has the form g_p and that the correspondence between $C_n(B(r_t(e)))$ and the nodes of g_p places $G(\phi_e^n(r_t(e)))$ in the left-most, i.e. 0th, position in the first row of $C_n(B(r_t(e)))$, for each $t<k$. [At most a relabelling is necessary to guarantee this.] Let B_i^t be the boy in the ith position of the top row of $C_n(B(r_t(e)))$ for each $i<T=(k-1)^{2p+1}k$ and each $t<k$. Let $G_0, G_1, \cdots, G_{T(k-1)-1}$ denote the first $T(k-1)$ female strangers. Introduce B_i^t to each of $G_i, G_{T+i}, G_{2T+i}, \cdots, G_{(k-2)T+i}$ for each $t<k$ and each $i<T$.

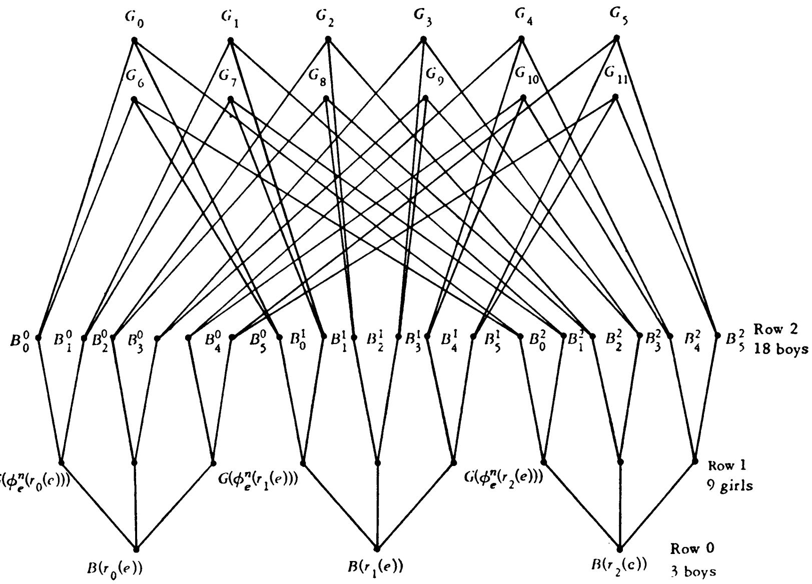
This completes the construction of Σ. Before we proceed to prove that it satisfies the desired properties we shall consider the following situation which contains within it the essence of the argument.

Suppose then that $k=3$ and that e is such that at stage $n=2^e$ we have that $\phi_e^n(r_t(e))$ is defined for each $t<3$ and that $B(r_t(e))$ knows $G(\phi_e^n(r_t(e)))$ for each $t<3$. At this stage each $C_n(B(r_t(e)))$ has the form g_0. After rearrangement these communities take the form



Thus the final case of the construction is the relevant one. After it is applied, we obtain the community $C_{n+1}(B(r_0(e)))$ which assumes the form below. It is now evident that in no solution to the marriage problem of $C_{n+1}(B(r_0(e)))$ can $B(r_t(e))$ marry $G(\phi_e^n(r_t(e)))$ for each $t<3$. For, of B_0^0, B_0^1, B_0^2 exactly two marry G_0 and G_6; the remaining one $B_0^{t^*}$ must marry $G(\phi_e^n(r_{t^*}(e)))$ so that $B(r_{t^*}(e))$ cannot marry her. Similarily, of B_1^0, B_1^1, B_1^2 exactly two marry G_1 and G_7; the remaining one $B_1^{t\#}$ must marry $G(\phi_e^n(r_t\#(e)))$ so that $B(r_t\#(e))$ cannot marry her. Hence, in fact, only (and exactly) one of $B(r_t(e))$ marries $G(\phi_e(r_t(e)))$.

We return now to the general case. It is clear from the construction that Σ is a recursive society (note that each introduction involves a stranger) and that each community of Σ is either finite, in which case each person



in it knows exactly k others, or has form g where g is the direct limit of the graphs g_p; so again each person in it knows exactly k other people. That Σ is symmetrically solvable follows from Lemma 1.

Thus we need only show that Σ is not recursively solvable—i.e. that no ϕ_e is a solution to the marriage problem of Σ. It suffices, of course, to show that if $\phi_e(r_t(e))$ is defined for each $t<k$ and $B(r_t(e))$ knows $G(\phi_e(r_t(e)))$ for each $t<k$ then no solution to the marriage problem of $C(B(r_0(e)))$ marries each $B(r_t(e))$ to the corresponding $G(\phi_e(r_t(e)))$. Note of course that under these hypotheses at some stage n we combine the $C_n(B(r_t(e)))$ into one community which is stable at stage n. We may assume that at stage n each $C_n(B(r_t(e)))$ has the form g_p (for some p) and that $G(\phi_e(r_t(e)))$ is in the leftmost position in the first row of $C_n(B(r_t(e)))$.

We shall show, by induction on $j<2p+3$ that for each i if A_t is the person in the ith position of the $(2p+3-j)$th row of $C_n(B(r_t(e)))$ for $t<k$ then in any solution to the marriage problem of $C_n(B(r_t(e)))$ exactly one of $\{A_t | t<k\}$ marries a person on the row below—i.e. on the $(2p+2-j)$th row. Thus taking $j=2p+2$ and $i=0$ we conclude that exactly one of $G(\phi_e(r_t(e)))$ marries $B(r_t(e))$, hence certainly not every $G(\phi_e(r_t(e)))$ marries $B(r_t(e))$.

For $j=0$ we must consider, for each fixed i, the boys $B_i^0, B_i^1, \cdots, B_i^{k-1}$ on the top row. Now at stage n each of these k boys was introduced to the $k-1$ girls $G_i, G_{T+i}, G_{2T+1}, \cdots, G_{(k-2)T+i}$. In any fixed solution to the marriage problem some B_i^t must marry a girl other than these; but the only other girl he knows is on the row below him. Also, since we added $[(k-1)^{2p+1}k]\cdot(k-1)$ girls at stage n, the total number of girls in $C_{n+1}(B(r_0(e)))$ is $[k+(k-1)^2k+\cdots+(k-1)^{2p}k]k+(k-1)^{2p+2}k$ which equals the total number of boys

$$[1+(k-1)k+(k-1)^3k+\cdots+(k-1)^{2p+1}k]k$$

in $C_{n+1}(B(r_0(e)))$, so that any solution to the marriage problem of $C_{n+1}(B(r_0(e)))$ is symmetric. Hence each G_{pT+i} must marry one of B_i^t. Hence exactly one B_i^t marries a girl on the row below him.

Now assume that the claim is proven for $j<2p+3$ and suppose that $j+1<2p+3$. Then each A_t in the ith position of the $(2p+2-j)$th row knows exactly $k-1$ people on the row above the $(2p+3-j)$th row, and these people are in the $(i(k-1))$th, $(i(k-1)+1)$th, $\cdots$, $(i(k-1)+(k-2))$th positions on the $(2p+3-j)$th row. Now exactly one of the people in the $(i(k-1)+s)$th position marries a person below him, for each $s<k-1$. Thus exactly $k-1$ of $A_0, A_1, \cdots, A_{k-1}$ marry people in the row above them. Hence exactly one of them marries a person in the row below. This completes the induction and the proof. □

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