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A NOTE ON THE RADON-NIKODYM THEOREM[1]

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ABSTRACT. This paper gives a necessary and sufficient condition in order that a bounded linear mapping from $L^1(\mu)$ into a Banach space be compact. It is applied to provide a slightly improved form of the Radon-Nikoydm theorem for vector valued measures and to give a sufficient condition in order that the range of a vector valued measure of bounded variation be compact.

In this paper, we give a sufficient condition in order that a measure is representable as an indefinite Bochner integral. We will show that a bounded linear operator T mapping $L^1(\mu)$ into a Banach space is compact if the set $\{T(\psi_{M_i}/\mu(M_i)):M_i \in \Sigma\}$ is precompact for every sequence of disjoint measurable sets $\{M_i\}$ (see Theorem 1). Combining this with a result of Rieffel [**2**] gives a slightly improved Radon-Nikodym theorem for the Bochner integral.

We apply this result in Theorem 2 to give a sufficient condition in order that the range of a vector valued measure be precompact. Here, we follow the approach of Uhl [**3**].

LEMMA. *Let (X, Σ, μ) be a finite measure space. Let $T:L^1(\mu)\to F$ be a bounded linear operator, where F is a Banach space. For each positive real number c, define $R(c)=\{T(\psi_M/\mu(M)):M \in \Sigma, 0<\mu(M)<c\}$ where ψ_M denotes the characteristic function of M. Then, $R(b)$ is a precompact set if and only if there is a number a such that $0<a<b$ and such that $R(a)$ is precompact.*

PROOF. Suppose that there is an a with $0<a<b$ such that $R(a)$ is precompact. Since the measure space is finite there are at most a finite number of atoms $A_1, A_2, \cdots, A_k$ in Σ with $a<\mu(A_i)$, $i=1, 2, \cdots, k$. Let $Q(a)=\{T(\psi_{A_i}/\mu(A_i)):i=1, 2, \cdots, k\}$ then $R(a)\cup Q(a)$ is precompact. Suppose that $y \in R(b)$, i.e. $y=T(\psi_M/\mu(M))$ for some $M \in \Sigma$ with $0<\mu(M)<b$. By Saks' theorem (Dunford-Schwartz [**1**, Lemma IV 9-7]) there exists a finite sequence of disjoint measurable sets $M_1, M_2, \cdots, M_n$ such that M_i

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is either an atom with $a<\mu(M_i)$ or $0<\mu(M_i)<a$ and $M=\bigcup_{i=1}^n M_i$. Hence,

$$y = T(\psi_M/\mu(M)) = \sum_{i=1}^n \frac{\mu(M_i)}{\mu(M)} T\left(\frac{\psi_{M_i}}{\mu(M_i)}\right)$$

is a member of the convex hull of the precompact set $R(a)\cup Q(a)$. Therefore the closure of $R(b)$ is a subset of the compact set $\mathrm{cl}[\mathrm{co}(R(a)\cup Q(a))]$, the closed convex hull of the set $R(a)\cup Q(a)$, and $R(b)$ is precompact.

THEOREM 1. *Let (X, Σ, μ) be a finite positive measure space and let F be a Banach space. Then a bounded linear operator $T: L^1(\mu)\to F$ is compact if the set $\{T(\psi_{M_i}/\mu(M_i)): M_i \in \Sigma\}$ is precompact for every sequence of disjoint measurable sets $\{M_i\}$.*

PROOF. We may assume that $\mu(X)=1$. It suffices to show that $R(1)$ is precompact since the image of the positive functions of the unit ball of $L^1(\mu)$ is the closed convex hull of $R(1)$. Indeed, if $\|f\|_1=1$ and $f>0$ then for a given $\varepsilon>0$ there is a simple function $\sum_{i=1} c_i\psi_{M_i}$, $c_i>0$, such that

$$\left\|\sum_{i=1}^n c_i\psi_{M_i}\right\| = 1 \quad \text{and} \quad \left\|f - \sum_{i=1}^n c_i\psi_{M_i}\right\|_1 < \varepsilon.$$

Now $\sum_{i=1}^n c_i\psi_{M_i}=\sum_{i=1}^n c_i\mu(M_i)(\psi_{M_i}/\mu(M_i))$ is a member of the convex hull of $R(1)$ since $\sum_{i=1}^n |c_i|\,|\mu(M_i)|=\|\sum c_i\psi_{M_i}\|_1=1$. By the preceding Lemma, it suffices to show that there exists an a with $0<a<1$ such that $R(a)$ is precompact. Suppose the contrary, i.e., none of $R(a)$ is precompact for each a with $0<a<1$; then there is an $\varepsilon>0$ such that none of $R(a)$ can be covered by a finite number of ε-balls $B(y_i, \varepsilon)=\{y \in F: \|y-y_i\|<\varepsilon\}$. Let $y_1 \in R(1)$ and by induction choose $y_n \in R(1/n)-\bigcup_{i=1}^{n-1} B(y_i, \varepsilon)$. The sequence $\{y_n\}$ is clearly infinite and has no convergent subsequence. Since $y_n \in R(1/n)$ there is a measurable set M_n such that

$$y_n = T(\psi_{M_n}/\mu(M_n)), \qquad n = 1, 2, \cdots,$$

and $\mu(M_n)<1/n$. Choose a subsequence $\{\alpha_i\}$ of $\{\mu(M_n)\}$ such that

$$\alpha_{i+1}/\alpha_i < 1/2^i, \qquad i = 1, 2, \cdots. \tag{1}$$

Now

$$\sum_{j>i}\alpha_j < \alpha_{i+1} + \frac{1}{2^{i+1}}\alpha_{i+1} + \frac{1}{2^{2i+3}}\alpha_{i+1} + \cdots < \frac{3}{2}\alpha_{i+1}, \tag{2}$$

Let $\alpha_i=\mu(M_{n(i)})$ and let $N_i=M_{n(i)}-\bigcup_{j>i} M_{n(j)}$. Clearly N_i and N_j are disjoint if $i \neq j$ and

$$\begin{aligned}
\mu(N_i) &\geqq \mu(M_{n(i)}) - \sum_{j>i} \mu(M_{n(j)}) \\
&\geqq \alpha_i - \frac{3}{2}\alpha_{i+1} \geqq \alpha_i - \frac{3}{2}\cdot\frac{1}{2^i}\alpha_i \quad \text{by (2) and (1)} \\
&= \alpha_i\left(1 - \frac{3}{2^{i+1}}\right) > 0, \qquad i = 1, 2, 3, \cdots,
\end{aligned}$$

and

$$\frac{\mu(N_i)}{\mu(M_{n(i)})} = \frac{\mu(N_i)}{\alpha_i} \geqq 1 - \frac{3}{2^{i+1}}, \qquad i = 1, 2, 3, \cdots. \tag{3}$$

Now

$$\begin{aligned}
&\left\| T\left(\frac{\psi_{N_i}}{\mu(N_i)}\right) - T\left(\frac{\psi_{M_{n(i)}}}{\mu(M_{n(i)})}\right)\right\| \leqq \|T\| \left\| \frac{\psi_{N_i}}{\mu(N_i)} - \frac{\psi_{M_{n(i)}}}{\mu(M_{n(i)})}\right\| \\
&\leqq \|T\| \left\{ \mu(N_i)\left(\frac{1}{\mu(N_i)} - \frac{1}{\mu(M_{n(i)})}\right) + \frac{\mu\left(\bigcup_{j>i} M_{n(j)}\right)}{\mu(M_{n(i)})} \right\} \\
&\leqq \|T\| \left\{ \frac{3}{2^{i+1}} + \frac{\frac{3}{2}\alpha_{i+1}}{\alpha_i} \right\} \quad \text{by (2) and (3).} \\
&= \frac{3}{2^i}\|T\| \to 0 \quad \text{as } i \to \infty.
\end{aligned}$$

Therefore we conclude that the sequence $\{T(\psi_{N_i}/\mu(N_i))\}$ has no convergent subsequence, which contradicts the hypothesis of the theorem.

REMARK. If the measure space (X, Σ, μ) is atom-free, the space need not be finite in order that the theorem hold.

COROLLARY. *Let (X, Σ, μ) be a σ-finite positive measure space. Let $\phi:\Sigma \to F$ be a vector valued measure of bounded variation, where F is a Banach space. Then ϕ is the indefinite integral with respect to μ of a Bochner integrable function $f: X \to F$ if and only if*

(1) *$\phi(M)=0$ whenever $\mu(M)=0$, where $M \in \Sigma$;*

(2) *ϕ has a finite total variation;*

(3) *given any $M \in \Sigma$ with $0<\mu(M)<\infty$, there is a set $N \in \Sigma$ so that $N \subseteq M$, $\mu(N)>0$ and N satisfies the following condition: if $\{N_i\}$ is any sequence of disjoint (nonnull) measurable sets in N, then $\{\phi(N_i)/\mu(N_i)\}$ is a precompact set.*

For the sufficiency, we apply Rieffel's theorem in [2] and note that the condition (3) given here implies the condition (3) of Rieffel, by using Theorem 1 above. (It is not difficult to check that by setting $T_N(S)=\phi(S)$, for each measurable set $S \subset N$, we obtain a bounded linear operator $T_N: L^1(N, \Sigma_N, \mu) \to F$ where Σ_N is the family of measurable sets contained in N. Indeed, $\|T_N(S)\| \leqq |\phi|(S)$ where $|\phi|$ is the total variation of ϕ. By the standard Radon-Nikodym Theorem, $|\phi|$ has a Radon-Nikodym derivative with respect to μ. Denoting this derivative by g, we get $\|T_N(S)\| \leqq \int_S |g| \, d\mu$. If we assume that g is bounded over N, as we may by replacing N with a suitable subset, if necessary, then $\|T_N(S)\| \leqq C \int_S d\mu$, where C is that bound.) The necessity of the condition is implied by Rieffel's theorem.

Let $\phi: \Sigma \to F$ be a vector valued measure of bounded variation and let μ be the total variation of ϕ, i.e., for each $M \in \Sigma$, $\mu(M) = \sup \sum_i \|\phi(M_i)\|$ where the sup is taken over all possible sequences of disjoint subsets M_i of M. It is shown in Uhl [3] that a sufficient condition in order that the range of ϕ be precompact is that the Banach space F be either a reflexive space or a separable dual space. In Uhl's proof the condition that the Banach space is either reflexive or separable dual is utilized to ensure that the measure ϕ admits a Bochner kernel representation, i.e., there is a Bochner integrable function f in $L^1(X, \Sigma, \mu)$ such that $\phi(M) = \int_M f \, d\mu$. We will apply the preceding theorem, using this approach of Uhl, to give a sufficient condition in order that the range of ϕ be precompact.

THEOREM 2. *If there exists a sequence $\{X^i\}$ of measurable subsets of X such that $\mu(X^i) < \infty$ for each i, $\mu(X - \bigcup_{i=1}^{\infty} X^i) = 0$ and such that for each i the set $\{\phi(M_k^i)/\mu(M_k^i);\ M_k^i \subset X^i$ and $M_k^i \in \Sigma\}$ is precompact for every sequence $\{M_k^i\}$ of disjoint measurable subsets of X^i, then the range of ϕ is precompact.*

PROOF. Let the operator $T: L^1(X, \Sigma, \mu) \to F$ be a linear extension of ϕ such that $T(\alpha\psi_M + \beta\psi_N) = \alpha\phi(M) + \beta\phi(N)$ for characteristic functions ψ_M, $M \in \Sigma$. Then the hypothesis of the theorem ensures that the restriction of the operator T to $L^1(X^i, \Sigma, \mu)$ is compact for each i (i.e., T is "locally compact"). Without loss of generality, we may assume that $\{X^i\}$ is a disjoint sequence of measurable sets. Therefore, by an inductive application of the Dunford-Pettis-Phillips theorem, there exists a Bochner integrable function $f: X \to F$ such that $T(g) = \int gf \, d\mu$ for each $g \in L^1(X, \Sigma, \mu)$. Now select a sequence $\{\psi_n\}$ of simple functions with their values in F such that $\lim_n \int \|f - \psi_n\| \, d\mu = 0$. Define $T: L^1(X, \Sigma, \mu) \to F$ by $T_n(g) = \int g\psi_n \, d\mu$ for each n. Then, by Hölder's inequality T_n is bounded for each n and $\lim_n \|T_n - T\| \leqq \lim_n \int \|f - \psi_n\| \, d\mu = 0$. Since ψ_n is simple, T_n is finite dimensional and hence is compact, thus, T also is compact. Since T is a linear extension of ϕ, the range of ϕ is precompact.

REMARK. This theorem extends Uhl's results. For, if the Banach space F is either a reflexive space or a separable dual space, then, by Dunford-Pettis-Phillips theorem, every bounded linear operator $T: L^1(X, \Sigma, \mu) \to F$

has a Bochner kernel; using Egorov's theorem it is seen that T is "locally compact" so that the hypothesis of the above theorem is satisfied. At the same time, the hypothesis of the theorem can be satisfied even when F is neither reflexive nor a separable dual, as we see in the following example.

EXAMPLE. Let $X=[0, \infty)$, $X_n=[n, n+1]$ and Σ be the σ-field of Borel sets of X and let λ denote the Lebesgue measure on X. Define a vector valued measure $\phi:\Sigma\to c_0$ by

$$\phi(M) = (\lambda(M \cap X_1), \tfrac{1}{2}\lambda(M \cap X_2), \cdots, (1/2^{n-1})\lambda(M \cap X_{n-1}), \cdots), \qquad M \in \Sigma.$$

Clearly ϕ is a measure with values in c_0 and $\|\phi\|\leqq 1$. The restriction of ϕ to X_n is $\phi|_{X_n}(M)=(0, 0, \cdots, (1/2^{n-1})(M\cap X_{n-1}), 0, \cdots)$ and it is compact. For any subset M of X_n, clearly $\phi(M)/\mu(M)=(0, 0, \cdots, 1, 0, \cdots)$ and $\{\phi(M)/\mu(M): M\subset X_n\}$ is compact.

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