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ON PERIODIC SOLUTIONS OF AUTONOMOUS HAMILTONIAN SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

DAVID C. CLARK

ABSTRACT. For the system $x''(t)+\operatorname{grad} U(x(t))=0$ lower bounds are obtained for the number of pairs $\pm x(t)$ of odd, periodic solutions, with the period prescribed. These bounds are in terms of the behavior of $U(x)$ near the origin and far away from the origin. It is assumed that $U(x)$ is even, and two different types of behavior of $U(x)$ far away from the origin are considered.

We seek to establish lower bounds for the number of odd solutions of a given period for the autonomous Hamiltonian system

$$x'' + \operatorname{grad} U(x) = 0,$$

where $x: R\to R^n$ and U is even and continuously differentiable. By a change of scale in the independent variable, we may normalize the problem so that we are seeking odd, 2π-periodic solutions of the system

$$\alpha^2 x'' + \operatorname{grad} U(x) = 0. \tag{1}$$

It is well known that if (1) is linear, there exist nontrivial solutions of (1) only for values of α in a certain discrete set. We shall be concerned with certain strictly nonlinear cases in which it turns out there are nontrivial solutions of (1) for values of α in a certain union of intervals. Thus it is reasonable to regard the value of α as given a priori.

Now we state some hypotheses and the main results.

(A1) $U(x)=\frac{1}{2}Bx\cdot x+\rho(x)$, for $x\in R^n$, where B is a constant symmetric matrix and $\rho(x)=o(|x|^2)$ as $|x|\to 0$.

(A2) $U(x)\leqq k_1\alpha^2|x|^2+k_2$, where $k_1<\frac{1}{2}$.

Let $\kappa_1\leqq\kappa_2\leqq\cdots\leqq\kappa_n$ be the eigenvalues of B.

THEOREM 1. *Let* (A1) *and* (A2) *hold, and let k be the number of pairs (i,j) of positive integers such that $i^2<\kappa_j/\alpha^2$. Then there are at least k distinct*

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pairs $\pm x(t)$ *of nonidentically zero solutions of* (1) *which are odd and have period* 2π.

(A3) $U(x)=\frac{1}{2}Cx\cdot x+\sigma(x)$, where C is a constant symmetric matrix and $|\text{grad }\sigma(x)|=o(|x|)$ as $|x|\to\infty$.

Let $\mu_1\leqq\mu_2\leqq\cdots\leqq\mu_n$ be the eigenvalues of C.

(A4) $\mu_j/\alpha^2\neq k^2$, $j=1,2,\cdots,n$; $k=1,2,3,\cdots$.

THEOREM 2. *Let* (A1), (A3), *and* (A4) *hold. Let k be as in Theorem* 1 *and let l be the number of pairs (i,j) of positive integers such that $i^2<\mu_j/\alpha^2$. If $k>l$, there are at least $k-l$ distinct pairs $\pm x(t)$ of nonidentically zero solutions of* (1) *which are odd and have period* 2π.

The existence of periodic solutions of (1) has been studied before by a number of authors ([**1**], [**2**], [**3**], [**6**]), most recently by Berger ([**1**], [**2**]), who has, among other results, obtained a generalization of a theorem of Lyapunov. The feature by which the results of the present paper differ from previous results is that the period of the solutions is regarded as fixed a priori.

PROOFS OF THEOREMS 1 AND 2. We shall reformulate the problem as one in the calculus of variations in a manner similar to that of Berger [**1**], and then we shall apply results of the author [**5**] on the Lusternik-Schnirelman theory of critical points of functionals.

Let H denote the Hilbert space of odd, 2π-periodic, absolutely continuous functions $x:R\to R^n$, such that $x'(t)$ is square integrable over $[0,2\pi]$, where the inner product and norm are

$$(x,y)=\int_0^{2\pi}x'(\ )\cdot y'(t)\,dt,\qquad \|x\|=(x,x)^{1/2}.$$

Let

$$f(x)=\tfrac{1}{2}\alpha^2\|x\|^2-\int_0^{2\pi}U(x(t))\,dt. \tag{2}$$

By standard regularity results, a critical point x of f relative to H satisfies the Euler equation

$$\alpha^2x''+\text{grad }U(x)=r, \tag{3}$$

where r is even. But since $x(t)$ is odd, the left side of (3) is odd in view of (A1); hence $r(t)=0$. Thus the critical points of f relative to H are odd, 2π-periodic solutions of (1).

With this reduction to a problem in the calculus of variations, Theorems 1 and 2 are consequences of results in [**5**]. Before stating these results, we need some definitions.

Let $\mathscr{H}$ denote a real Hilbert space, $\phi(v)$ a real valued function on $\mathscr{H}$

with a continuous Fréchet derivative $\phi'(v)$. The following condition is a variant of the Palais-Smale condition:

(C) *Every bounded sequence* $\{v_k\}\subset\mathscr{H}$ *such that* $\{\phi(v_k)\}$ *is bounded and* $\phi'(v_k)\to 0$ *contains a convergent subsequence* $\{v_{k_l}\}$.

Let q be a quadratic form on $\mathscr{H}$. If k is the maximal dimension of subspaces of $\mathscr{H}$ on which q is negative definite, we say that q is of *index* k. ($k=\infty$ is allowed.) q is called *regular* if $q'(v)$ exists and there exists $a>0$ such that $\|q'(v)\|\geqq a\|v\|$ for all $v\in\mathscr{H}$.

THEOREM A. *Let $\phi(v)$ be a real valued function on a real Hilbert space $\mathscr{H}$ such that ϕ is even, C^1, satisfies condition* (C), *is bounded below, $\phi(v)\geqq 0$ for large $\|v\|$ and $\phi(v)=q(v)+o(\|v\|^2)$ as $\|v\|\to 0$, where q is a quadratic form of index k. Then ϕ has at least k pairs $\pm v$ of nonzero critical points.*

THEOREM B. *Let ϕ be a real valued function on a real Hilbert space $\mathscr{H}$ such that ϕ is even, C^1, satisfies condition* (C), *is bounded below on bounded sets, $\phi(v)=q_1(v)+o(\|v\|^2)$ as $\|v\|\to 0$, where q_1 is a quadratic form of index k, and $\phi(v)=q_2(v)+r(x)$, where q_2 is a continuous, regular quadratic form of finite index l, and where $\|r'(v)\|=o(\|v\|)$ as $\|v\|\to\infty$. If $k>l$, then ϕ has at least $k-l$ pairs $\pm v$ of nonzero critical points.*

We now begin verifying the hypotheses of Theorems A and B for the particular cases involved in Theorems 1 and 2.

Since U is even, f is even. Since $U\in C^1(R^n)$, it is easily shown that f has a locally uniform Fréchet derivative f' which is locally bounded. Hence f' is continuous [7, p. 43] and f is C^1.

Also, since

$$\sup_{0\leqq t\leqq 2\pi}|x(t)|\leqq \text{const}\,\|x\|,\quad \text{if } x\in H, \tag{4}$$

and since $U\in C^1(R^n)$, f is bounded on bounded sets in H.

LEMMA 1. *f satisfies condition* (C) *relative to H.*

PROOF. The proof is suggested by ideas of Browder, e.g. [**4**]. The only hypothesis used is that $U\in C^1(R^n)$. The Fréchet derivative $f'(x)$ satisfies

$$(f'(x), y)=\alpha^2(x, y)-\int_0^{2\pi}\operatorname{grad} U(x(t))\cdot y(t)\,dt. \tag{5}$$

Let $\{x^{(p)}\}\subset H$ be bounded and such that $f'(x^{(p)})\to 0$. It suffices to show that a subsequence of $\{x^{(p)}\}$ converges strongly. Since $\{x^{(p)}\}$ is bounded, a subsequence of $\{x^{(p)}\}$ converges weakly. We may assume that $\{x^{(p)}\}$ itself converges weakly. Denote the weak limit by x. We have that

$$\lim_{p\to\infty}(f'(x^{(p)})-f'(x), x^{(p)}-x)=0, \tag{6}$$

since $f'(x^{(p)})\to 0$ and $x^{(p)}\rightharpoonup x$. On the other hand, by (5),

$$(7)\quad \begin{aligned}(f'(x^{(p)}) - f'(x), x^{(p)} - x) &= \alpha^2 \|x^{(p)} - x\|^2 \\ &\quad - \int_0^{2\pi} \operatorname{grad} U(x^{(p)}(t)) - \operatorname{grad} U(x(t)) \cdot (x^{(p)}(t) - x(t))\, dt.\end{aligned}$$

Since $x^{(p)}\rightharpoonup x$, $x^{(p)}(t)\to x(t)$ uniformly; hence by (7),

$$\lim_{p\to\infty} (f'(x^{(p)}) - f'(x), x^{(p)} - x) = \alpha^2 \lim_{p\to\infty} \|x^{(p)} - x\|^2.$$

Hence, by (6), $\lim_{p\to\infty}\|x^{(p)}-x\|=0$, and so $\{x^{(p)}\}$ converges strongly. This completes the proof.

LEMMA 2. *Under assumption* (A1), $f(x)=q_1(x)+o(\|x\|^2)$ *as* $\|x\|\to 0$, *where* $q_1(x)$ *is a quadratic form of index k equal to the number of pairs* (i,j) *of positive integers such that* $i^2<\kappa_j/\alpha^2$.

PROOF. By (A1) and (2), we may write $f(x)=q_1(x)+r_1(x)$, where

$$q_1(x) = \tfrac{1}{2}\alpha^2 \|x\|^2 - \tfrac{1}{2}\int_0^{2\pi} Bx(t)\cdot x(t)\, dt, \qquad r_1(x) = -\int_0^{2\pi} \rho(x(t))\, dt.$$

By (A1) and (4), $r_1(x)=o(\|x\|^2)$ as $\|x\|\to 0$.

It remains to calculate the index of q_1. Let $u_1, u_2, \cdots, u_n$ be an orthonormal set of eigenvectors of B corresponding to $\kappa_1, \kappa_2, \cdots, \kappa_n$, respectively. Let H_0 be the subspace of H spanned by the functions $\sin(it)u_j$, where $i^2<\kappa_j/\alpha^2$. Then it is easily shown that q_1 is negative definite on H_0 and positive semidefinite on $H_0^{\perp}$. Hence the index of q_1 is the dimension of H_0, which is the value k defined in the lemma.

LEMMA 3. *Under assumption* (A2), $f(x)\geqq 0$ *for large* $\|x\|$.

PROOF. By expansion of elements $x(t)$ of H in Fourier sine series, it can be shown that

$$(8)\qquad \|x\|_{L_2} \leqq \|x\|.$$

Hence, by (A2), $f(x)\geqq(\frac{1}{2}-k_1)\alpha^2\|x\|^2-k_2 2\pi$, so $f(x)\geqq 0$ for large $\|x\|$.

From Lemma 3 and the preceding remark that f is bounded on bounded sets, it follows that if (A2) holds, f is bounded below.

Theorem 1 follows from Theorem A, Lemmas 1 to 3, and the preceding remarks in this section.

Under assumption (A3) we may write $f(x)=q_2(x)+r_2(x)$, where

$$q_2(x) = \tfrac{1}{2}\alpha^2 \|x\|^2 - \tfrac{1}{2}\int_0^{2\pi} Cx(t)\cdot x(t)\, dt, \qquad r_2(x) = -\int_0^{2\pi} \sigma(x(t))\, dt.$$

LEMMA 4. *The quadratic form $q_2(x)$ is continuous in H, and has index l equal to the number of pairs (i, j) of positive integers such that $i^2 < \mu_j/\alpha^2$. Under assumption* (A4), *$q_2(x)$ is regular.*

PROOF. Clearly $q_2(x)$ is bounded on the unit sphere in H, hence q_2 is continuous in H.

Let $u_1, u_2, \cdots, u_n$ be an orthonormal set of eigenvectors corresponding to $\mu_1, \mu_2, \cdots, \mu_n$ respectively. Let H_0 be the subspace of H spanned by $\sin(it)u_j$ such that $i^2 < \mu_j/\alpha^2$. Then $H_0^\perp$ is spanned by $\sin(it)u_j$ such that $i^2 > \mu_j/\alpha^2$ in view of (A4). Also q_2 is negative definite on H_0 and positive definite on $H_0^\perp$. The dimension of H_0 is l, and is clearly the index of q_2.

We have that $q_2(x) = \frac{1}{2}\beta(x, x)$, where

$$(x, y) = \alpha^2(x, y) - \int_0^{2\pi} Cx(t) \cdot y(t)\, dt.$$

β is a symmetric bilinear form. Also, $(q_2'(x), y) = \beta(x, y)$. Hence

$$\|q_2'(x)\| \geqq |\beta(x, y)|/\|y\|. \tag{9}$$

It is easily shown that

$$\beta(u, v) = 0 \quad \text{if } u \in H_0,\ v \in H_0^\perp. \tag{10}$$

For any $x \in H$, we write $x = u + v$, $u \in H_0$, $v \in H_0^\perp$. Using (9) and (10),

$$\begin{aligned} \|q_2'(x)\| &\geqq |\beta(u, u)|/\|u\| = 2\,|q_2(u)|/\|u\|, \\ \|q_2'(x)\| &\geqq |\beta(v, v)|/\|v\| = 2\,|q_2(v)|/\|v\|, \end{aligned} \tag{11}$$

if $u \neq 0$ and $v \neq 0$. It is clear that $|q_2(u)|/\|u\|^2$ and $|q_2(v)|/\|v\|^2$ are bounded below for $0 \neq u \in H_0$ and $0 \neq v \in H_0^\perp$. Hence, from (11),

$$\|q_2'(x)\| \geqq \max(c\|u\|, c\|v\|) \tag{12}$$

for some $c > 0$. Since either $\|u\| \geqq \frac{1}{2}\|x\|$ or $\|v\| \geqq \frac{1}{2}\|x\|$, it follows from (12) that q_2 is regular.

LEMMA 5. *Under assumption* (A3), $r_2'(x) = o(\|x\|)$ *as* $\|x\| \to \infty$.

PROOF. We have the equality

$$(r_2'(x), y) = -\int_0^{2\pi} \operatorname{grad} \sigma(x(t)) \cdot y(t)\, dt.$$

Setting $y = r_2'(x)$ and using inequality (8),

$$\|r_2'(x)\| \leqq \|\operatorname{grad} \sigma(x(t))\|_{L_2}. \tag{13}$$

It follows from (A3) that $\|\operatorname{grad} \sigma(x(t))\|_{L_2} = o(\|x\|_{L_2})$ as $\|x\|_{L_2} \to \infty$. Hence, from (8) and (13), $\|r_2'(x)\| = o(\|x\|)$ as $\|x\| \to \infty$.

Theorem 2 follows from Theorem B, Lemmas 1, 2, 4, and 5, and the remarks in this section.

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Department of Mathematics, University of Puerto Rico, Mayaguez, Puerto Rico 00708