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A CASE IN WHICH IRREDUCIBILITY OF AN ANALYTIC GERM IMPLIES IRREDUCIBILITY OF THE TANGENT CONE[1]

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ABSTRACT. There are simple examples in which a variety is irreducible at a point but has a reducible tangent cone. The following theorem is proved. If X_p is an irreducible analytic germ and if the Jacobian ideal beomes principal on the normalization then the tangent cone of X at p is irreducible. If, moreover, the singular set of X is a manifold at p then X is Whitney a,b-regular along the singular set at p.

Let X be a pure r-dimensional analytic subset of open U in C^n. $\mathcal{O}^n$ denotes the holomorphic structure sheaf on U, I denotes the (self-radical) ideal sheaf defining X in U, and $\mathcal{O}$ denotes the resulting (reduced) holomorphic structure on X. Let $T=(T_1, \cdots, T_n)$ be coordinates on C^n. Hereafter J denotes the Jacobian ideal of X on X. It is a coherent sheaf of ideals whose stalk at p is obtained as follows. Take the $\infty \times n$ matrix whose entries are of the form $(\partial f/\partial T_j|_X, j=1, \cdots, n, f\in I_p)$. J_p is the ideal in $\mathcal{O}_p$ generated by the $(n-r)\times(n-r)$ subdeterminants of this matrix. It is sufficient to restrict f to a system of generators for I_p in which case one obtains a system of generators for J_p. Thus we can assume that f belongs to a finite set $\mathcal{F}$ of generators for I_p. By replacing U by a smaller open set we may also assume f is holomorphic on U, $\forall f\in\mathcal{F}$, $\{f_q, f\in\mathcal{F}\}$ generates I_q, and the $(n-r)\times(n-r)$ subdeterminants obtained from the matrix

$$(*)\qquad (\partial f/\partial T_j|_X, j = 1, \cdots, n, f\in\mathcal{F})$$

have germs at q which generate the ideal J_q for all q in X. Let $t=\#(\mathcal{F})$.

The singular set of X, hereafter denoted $Sg(X)$, is the locus of J.

Let $\pi:(X', \mathcal{O}')\to(X, \mathcal{O})$ denote the normalization of X. The following are

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in natural one-to-one correspondence: $\{p_1', \cdots, p_k'\}=\pi^{-1}(p)$, $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_k\}=$ minimal primes of $\mathcal{O}_p$, $\{X_1, \cdots, X_k\}=$ irreducible components of the germ of X at p. We say $J\mathcal{O}'$ is locally principal over p if and only if $J_p\mathcal{O}'_{p_\lambda'}$ is principal for $\lambda=1, \cdots, k$. This is an open condition on X.

Let $T(X)=\{(p,u)\in X\times C^n \mid \sum f_j(p)u_j=0\ \forall f\in\mathcal{F}\}$ where $f_j=\partial f/\partial T_j$. $T(X,p)$ denotes the fiber of the projection of $T(X)$ to X. The singular points of X are those for which $\dim T(X,p)>r$. We let Y denote the set of singular points. $C_4(X)$ is the closure in $T(X)$ of $T(X-Y)$. $C_4(X,p)$ is the fiber of $C_4(X)$ over p. It is the union of a collection of r-dimensional subspaces of C^n. Let $G^{n-1,r-1}$ be the Grassmann variety of r-dimensional subspaces of C^n. If q is simple on X, $T(X,q)$ has a unique point $\tau(X,q)$ in $G^{n-1,r-1}$. Let $\tau(X)$ denote the closure of $\{(q,\tau(X,q)) \mid q\in X-Y\}$ in $X\times G^{n-1,r-1}$. Let $\tau(X,p)$ denote the fiber of the projection to X. See [**6**, Theorem 5.1, p. 218 and Theorem 7.1, p. 224]. There is a natural map $\varphi: C_4(X)\to\tau(X)$ which commutes with the projections to X and makes $C_4(X)$ a fiber space of constant fiber dimension $n-r$ over $\tau(X)$. $\tau(X)\to X$ is a proper modification of X which is an isomorphism exactly over $X-Y$. Denote the structure sheaf on $\tau(X)$ by $\tilde{\mathcal{O}}$.

Let $(B_J(X), \hat{\mathcal{O}})$ denote the blowing-up of X along J. This is a subspace of $X\times P^N$ which can be realized as follows. Let $\mathcal{J}$ denote the functions on X obtained from (*). These are determinants which we enumerate in the following way for our subsequent convenience. Let $\mathcal{F}=(f_1, \cdots, f_t)$ and let $\lambda=(\lambda_1, \cdots, \lambda_{n-r})$. By J_λ we mean the tuple obtained from (*) by taking subdeterminants from the $(n-r)\times n$ submatrix of (*) whose rows involve the functions f_{λ_j}, $j=1, \cdots, n-r$. Let J_λ^k, $k=1, \cdots, \binom{n}{n-r}$ denote the entry of this tuple ordered in some fixed manner. Let $\mathcal{J}=(\cdots, J_\lambda, \cdots)$ be a fixed ordering of the tuples J_λ. Introduce indeterminants Z_λ^k ordered in the same manner. Then $(B_J(X), \hat{\mathcal{O}})$ is the analytic space defined by the equations $\{Z_\lambda^k J_\mu^j - Z_\mu^j J_\lambda^k=0$, all λ, μ, k, j subject to the above conventions$\}$ in $X\times P^{N-1}$ where the Z_λ^k are homogeneous coordinates on P^{N-1} and $N=\binom{t}{n-1}\binom{n}{n-r}$. $B_J(X)\to X$ blows up exactly over $X-Y$. We denote the fiber of this projection by $B_J(X,p)$.

The tangent cone to X at p, denoted $C(X,p)$, is $\{u\in C^n \mid f^*(u)=0\ \forall f\in I_p\}$ where f^* denotes the leading form at p of f. Since $G(\mathcal{O}_p)$, the associated graded ring of $\mathcal{O}_p$ with respect to its maximal ideal, is isomorphic to $C[T]/A$ where A is the ideal generated by $\{f^*(T) \mid f\in I_p\}$, there is a one-to-one correspondence between the components of the cone $C(X,p)$ and the minimal primes of $G(\mathcal{O}_p)$. We use this and the fact, [**6**, (3.1), p. 212], that $C_4(X,p)\supset C(X,p)$ to prove the theorem of this paper.

PROPOSITION 1. *With the notation and assumptions of the preceding paragraphs, the following are equivalent*:

(i) $\dim C_4(X,p)=r$,
(ii) $\tau(X,p)$ *is finite*,
(iii) $B_J(X,p)$ *is finite*,
(iv) J *is locally principal over* p *on* $(X', \mathcal{O}')$,
(v) $C_4(X,p)=C(X,p)$.

Moreover, if these conditions hold then $\#\tau(X,p)\leqq\#B_J(X,p)\leqq\#\{$*irreducible components of* $X_p\}$ *and* $C(X,p)$ *is a union of* $\#\tau(X,p)$ r-*dimensional subspaces of* C^n.

PROOF. As remarked earlier $C_4(X,p)$ is the union of r-dimensional subspaces of C^n. $\tau(X,p)$ is the union of the Grassmann coordinates of these subspaces. Both $C_4(X,p)$ and $\tau(X,p)$ are algebraic sets. Clearly $\dim C_4(X,p)=r$ if and only if $\tau(X,p)$ is finite which proves the equivalence of (i) and (ii).

Indices have the meanings established in the description of $B_J(X)$. We will show that to each point of $\tau(X, p)$ there corresponds at least one and at most finitely many points of $B_J(X,p)$. (This is always true. Moreover, when X is a complete intersection $(\tau(X), \tilde{\mathcal{O}})$ and $(B_J(X), \hat{\mathcal{O}}\text{red})$ are isomorphic. In [**4**, Remark (1.2), p. 3] it seems to be stated that they are always isomorphic. This does not appear to be true to us although we have been unable to produce a counterexample.) Let $(p, \alpha)\in\tau(X)$. Let a be affine coordinates for α. There exist simple points $p_\nu\to p$, $c_\nu\in C$, such that $c_\nu J_\mu(p_\nu)\to a$ for at least one μ. Moreover, given any μ such that $J_\mu(p_\nu)\neq 0$ for almost all ν a sequence d_ν exists such that $d_\nu J_\mu(p_\nu)\to a$ on the subsequence of $\{p_\nu\}$ at which $J_\mu(p_\nu)\neq 0$. The reason for this is that, if $J_\mu(p_\nu)\neq 0$, then J_μ is (on the ray corresponding to) the Grassmann coordinates of $T(X,p_\nu)$. Consequently $c_\nu(\cdots, J_\lambda(p_\nu), \cdots)$ converges (on a subsequence of $\{p_\nu\}$) to a point $b=(\cdots, b_\lambda, \cdots)$ with the following properties (where β denotes the point of P^{N-1} corresponding to b): $(p,\beta)\in B_J(X)$; if $b_\lambda\neq 0$ then b_λ is proportional to a; $b_\lambda\neq 0$ for some λ. Thus, given (p, α) in $\tau(X)$ we have constructed a point (p,β) in $B_J(X)$. How many such points can exist? The construction depends not only on (p, α) but also on the sequence $\{p_\nu\}$. It is conceivable that for a different sequence $\{q_\nu\}$, $\{\lambda|b_\lambda\neq 0\}$ could change yielding another point of $B_J(X,p)$ corresponding to (p, α). Since the only possible variation is in $\{\lambda|b_\lambda\neq 0\}$ and since there are only finitely many choices for the tuple λ, we conclude there are at most finitely many points on $B_J(X,p)$ for each point on $\tau(X,p)$. Consequently (ii) and (iii) are equivalent and the first inequality is proved.

Suppose $B_J(X,p)$ is finite. Since $B_J(X)\to X$ is a proper modification, $B_J(X,q)$ is finite for all q near p and there is a natural map $\theta:(X', \mathcal{O}')\to(B_J(X), \hat{\mathcal{O}})$ commuting with the projections to X over a neighborhood of p. θ is proper and surjective. (This is the universal mapping property of normalization [**1**, 46.20, p. 456].) $J\hat{\mathcal{O}}$ is locally principal and $J\mathcal{O}'$ is the pull-back of $J\hat{\mathcal{O}}$ induced by θ so $J\mathcal{O}'$ is locally principal which proves (iii) implies (iv) and the second inequality.

We also use θ to prove (iii) implies (v). Both $C_4(\ ,p)$ and $C(\ ,p)$ distribute across unions so we may assume X_p is irreducible and only one point p' of X' lies over p. Hence, existence of θ shows that only one point is in $B_J(X,p)$. Consequently $\tau(X,p)$ consists of one point. Hence, $C_4(X,p)$ consists of a single plane of dimension r. Since every component of $C(X,p)$ has dimension r (proving (v) implies (iii)) and $C_4(X,p)\supset C(X,p)$, it follows that they are equal. This proves that (iii) implies (v).

It remains to prove that (iv) implies (iii). If $J\mathcal{O}'$ is locally principal over p then there is a natural map $\psi\colon(X',\mathcal{O}')\to(B_j(X),\hat{\mathcal{O}})$, which commutes with projections to X over a neighborhood of p. ψ is proper and surjective. (This is the universal mapping property of blowing-up [**3**, p. 123].) Since X' has finite fiber over p, $B_J(X,p)$ is finite. Q.E.D.

COROLLARY. *Suppose X_j denote the irreducible components of X at p, $I_p(X)$, $I_p(X_j)$, is the ideal defining X, resp. X_j, at p and J, J_j, is the Jacobian ideal defined by $I_p(X)$, resp. $I_p(X_j)$. Then J becomes locally principal over p on the normalization of X if and only if J_j becomes principal over p on the normalization of X_j for all j.*

PROOF. $C_4(X,p)=\bigcup C_4(X_j,p)$. Q.E.D.

THEOREM 2. *Let R be the local ring of a point p on a reduced, pure r-dimensional analytic space X. Suppose that the Jacobian ideal becomes locally principal over p on the normalization of X. Then each minimal prime ideal P of $G(R)$ determines at least one minimal prime ideal $\mathfrak{p}$ of R such that P is the radical of* $\mathrm{Ker}(G(R)\to G(R/\mathfrak{p}))$. *Moreover, $G(R)/P$ is a polynomial ring in r variables over C.*

PROOF. The hypothesis implies that $C(X,p)=C_4(X,p)$ and is a finite union of r-dimensional subspaces of C^n. These planes are in one-to-one correspondence with the minimal primes of $G(R)$. Let L be the plane corresponding to P. Because $C(\ ,p)$ distributes across unions, there is at least one irreducible component Z of X_p such that $L\subset C(Z,p)$. Applying the inequality of Proposition 1 we conclude that $\#\tau(Z,p)=1$ and $L=C(Z,p)$. Z is determined by a minimal prime ideal of R and P is the radical of the kernel of $G(R)\to G(R/\mathfrak{p})$. $G(R/\mathfrak{p})$ has as reduction a ring of polynomials in r indeterminants because L is a plane of dimension r. Since $G(R)\to G(R/\mathfrak{p})$ is surjective $G(R)/P$ is isomorphic to the reduction of $G(R/\mathfrak{p})$. Q.E.D.

COROLLARY. *If, in addition to the hypothesis of the theorem, R is a domain, then $G(R)$ has as reduction an integral domain.*

PROOF. (0) is the only minimal prime of R so $G(R)$ can have only one minimal prime. Q.E.D.

EXAMPLES. The locus of $Y^2-X^3=0$ is an example showing that $G(R)$ need not be a domain.

In general the correspondence between primes of $G(R)$ and primes of R is one-to-many, e.g. the locus of $Y(Y-X^2)=0$.

The locus of $XY-Z^3=0$ shows that assuming $C(X,p)$ a union of planes and X_p irreducible does not insure that $C(X,p)$ is irreducible.

Question. What condition together with irreducibility of X_p insures irreducibility of $C(X,p)$? The condition given here seems overly strong since it insures that $C(X,p)$ is a plane, not just irreducible.

Recall the Whitney conditions [**6**, §8]. If X is an analytic space, Y is a manifold, and $p \in X\cap Y$, X is said to be a-regular along Y at p if: whenever $\{p_\nu\} \in X-Sg(X)$ with $p_\nu \to p$ and $T(X,p_\nu)\to T$ then $T\supset T(Y,p)$. X is said to be b-regular along Y at p if: whenever $\{p_\nu\}\in X-Sg(X)$, $\{q_\nu\}\in Y$, $\{c_\nu\}\in C$ with $p_\nu\to p$, $q_\nu\to p$, $T(X,p_\nu)\to T$ and $c_\nu(p_\nu-q_\nu)\to v$ then $v\in T$. X is said to be a,b-regular along Y at p if X is both a-regular and b-regular along Y at p. We can see that $J\mathcal{O}'$ locally principal over p allows us to determine $\lim T(X,p_\nu)=T$. For the limit to exist, infinitely many p_ν must be on one component X_j of X at p and $T=C(X_j,p)$. Thus a-regularity reduces to the simple property: every component of $C(X,p)$ contains $T(Y,p)$ at p if $J\mathcal{O}'$ is locally principal over p. This suggests that we examine the question: Does $J\mathcal{O}'$ locally principal over p insure X is a,b-regular along $Sg(X)$ at p?

First we observe that $J\mathcal{O}'$ locally principal over p implies that either $p\notin Sg(X)$ or else $p\in Sg(X)$ and $\dim_p Sg(X)=r-1$. This is because the locus of J is π (locus of $J\mathcal{O}'$). The latter either has dimension $r-1$ at some point over p or is empty at every point over p and π preserves dimensions.

In what follows we sometimes require that $Sg(X)$ be a manifold at p. By this we mean that it is a manifold with the reduced structure, not that it is a manifold with the structure induced by J.

A fundamental tool is the following proposition proved by John Stutz.

PROPOSITION 3. *Let X be a reduced analytic space of pure dimension r at p with p in $Sg(X)$ and $Sg(X)$ a manifold at p. Assume that* $\dim C_4(X,p)=r$. *Then X has a Puiseux series normalization at p, i.e. there exists a ball $D\subset C^r$ and holomorphic maps $f_j\colon D\to X_j$ where X_j are the irreducible components of X at p such that*

(a) *f_j is a homeomorphism;*

(b) *there are coordinates (x), (y) in C^r and C^m (the ambient space for X at* *p) so that* $y(p)=0$ *and* $f_j(x)=(x_1, \cdots, x_{r-1}, x_r^{d_j}, f_{r+1,j}(x), \cdots, f_{m,j}(x))$, *and the ball* D *and coordinates* (x), (y) *are independent of* j; *and*

(c) $Y=C_{y_1,\cdots,y_{r-1}}$;

(d) *if* X_j *contains* $Sg(X)$ *at* p *then* $d_j \leqq$ *order of* $f_{k,j}(x)$ *in* x_r $\forall k$.

PROOF. [**5**, Propositions 4.2 and 4.6].

COROLLARY. *With* X *as in the proposition the normalization of* X *is a manifold.*

EXAMPLE. The locus X of $y^2-x^2(z-x)$ has singular set C_z and normalization a manifold but $C_4(X, 0)$ has dimension three so the condition is not necessary.

THEOREM 4. *Let* X *be a reduced analytic space of pure dimension* r *at* p *with* p *in* $Sg(X)$ *and* $Sg(X)$ *a manifold at* p. *Suppose that the Jacobian ideal of* X *becomes locally principal over* p *on the normalization of* X. *Then* X *is* a,b*-regular along* $Sg(X)$ *at* p *if and only if every irreducible component of* X *at* p *contains* $Sg(X)$ *at* p. *Any component of* X *which does not contain* $Sg(X)$ *at* p *is a manifold at* p.

PROOF. (We understand that Stutz has also proved this theorem in a paper of his to appear in the Amer. J. Math. so we give only an indication of the proof.) Using the Jacobian matrix of the mapping f_j of Proposition 3 it is easy to see that $Sg(X_j) \subset Y$. If X_j does not contain Y, dim $Sg(X_j) < r-1$. By an earlier remark $Sg(X_j)=\varnothing$ which proves the last claim of the theorem. Now the only if part follows from Hironaka's result [**2**, Corollary 6.2] that a,b-regularity implies equimultiplicity. The converse follows by careful analysis of the Jacobian matrix of the mapping f_j of Proposition 3. Q.E.D.

EXAMPLES. It is easy to use this theorem to construct an example in which J principalizes on the normalization but X is not a,b-regular along $Sg(X)$ at p. Zariski has given an example in which J becomes principal on the normalization but $Sg(X)$ is not manifold at p [**7**, footnote 3, p. 987].

REMARKS. In [**5**] Stutz proved, among other things, a number of the results of this paper under additional hypotheses. He proved the equivalence of (i) and (v) of Proposition 1 assuming that $Sg(X)$ is a manifold of dimension $r-1$ at p and dim $C_5(X,p)=r+1$. He proved Theorem 4 assuming p simple in $Sg(X)$, $Sg(X)$ a manifold of dimension r at p, dim $C_4(X,p)=r$, and dim $C_5(X,p)=r+1$. We improve upon this result by applying Proposition 3 to the question of a, b-regularity directly rather than passing through the existence of wings as Stutz did. The assumptions used by Stutz insure that every component of X at p contains $Sg(X)$ at p but he has told us the converse is not true. Where this paper extends part of [5] most effectively is in the characterization of dim $C_4(X, p)=r$ by means of the Jacobian ideal (Zariski uses this technique to get a criterion of equisingularity in [7, Theorem 5.1, p. 987].) Not only does this technique yield better results, e.g. dim $C_4(X, p)=r$ always implies $C_4(X, p)=C(X, p)$ and an avoidance in Theorem 4 of the cone $C_5(X, p)$ which is difficult to compute, but it raises a number of interesting questions in the formal case.

Question. Suppose $R=S/\mathfrak{p}$ where $\mathfrak{p}$ is prime and $S=k[[y_1, \cdots, y_n]]$ and R/J is regular and JR' is principal where R' is the integral closure of R. Does it follow that R has a Puiseux series normalization? Is R' regular? Is the reduction of $G(R)$ a domain? These results are true in the convergent case, yet the hypothesis and conclusion are punctual but the proofs are not. We hope to turn to these questions in a subsequent paper.

We are indebted to Abhyankar who proposed the question: What are the consequences of J locally principal over p on the normalization of X? and to the referee who suggested we examine the relation between some of these results and the paper of Stutz.

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