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THE SCHNIRELMANN DENSITY OF THE $(k, 2)$-INTEGERS

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ABSTRACT. The Schnirelmann density of the $(k, 2)$-integers is found for each $k \geqq 4$.

1. **Introduction.** Let the integers k, r satisfy $0<r<k$. Then an integer of the form $a^k b$, where a is an integer and b an r-free integer, is called a (k, r)-integer. The (k, r)-integers were introduced by Subbarao and Harris [4] and their properties studied by Subbarao and Feng [3] and Feng and Subbarao [1]. We denote the set of positive (k, r)-integers by $Q_{k,r}$ and the set of positive r-free integers by Q_r. If Q is a set of positive integers then the number of integers $\leqq x$ in Q is denoted by $Q(x)$. The Schnirelmann density of Q is $D(Q)=\inf_{n\geqq 1}(Q(n)/n)$. Here, and throughout this note, n denotes a positive integer.

K. Rogers [2] proved the following:

THEOREM (ROGERS). *$D(Q_2)=53/88$ and $Q_2(n)/n=53/88$ if and only if $n=176$.*

In this note we prove the following:

THEOREM. (i) *If $k\geqq 8$ then $D(Q_{k,2})=53/88$ and $Q_{k,2}(n)/n=53/88$ if and only if $n=176$.* (ii) *If $k=6$ or 7 then $D(Q_{k,2})=17/28$ and $Q_{k,2}(n)/n=17/28$ if and only if $n=28$ or 56.* (iii) *$D(Q_{5,2})=17/28$ and $Q_{5,2}(n)/n=17/28$ if and only if $n=28$.* (iv) *$D(Q_{4,2})=9/14$ and $Q_{4,2}(n)/n=9/14$ if and only if $n=28$.*

The values of $D(Q_{3,2})$, $D(Q_r)$ and $D(Q_{k,r})$ for $k>r\geqq 3$ are not known (see [1]).

2. **Proof of (i).** We have

$$Q_{k,2}(n) = \sum_{s=1}^{\infty} Q_2(n/s^k). \tag{1}$$

Hence $Q_{k,2}(n)=Q_2(n)$ if $n<2^k$ and $Q_{k,2}(n)>Q_2(n)$ if $n\geqq 2^k$. Hence (i) follows from Rogers's theorem since $176<2^k$ if $k\geqq 8$.

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3. **Proof of (ii) for $k=7$.** From (1) and Rogers's theorem,

$$\begin{aligned} Q_{7,2}(n) &\geqq Q_2(n) + Q_2(n/2^7) + Q_2(n/3^7) \\ &\geqq \frac{53}{88}\left(n + \frac{n-127}{128} + \frac{n-2186}{2187}\right) \qquad (2) \\ &> \frac{17}{28} n \quad \text{if } n \geqq 10855. \end{aligned}$$

It therefore remains to prove the result (ii) for $n \leqq 10854$, which we assume in what follows.

From (2) and Rogers's theorem,

$$Q_{7,2}(n) \geqq \frac{53}{88} n + Q_2\left(\frac{m}{128}\right) + Q_2\left(\frac{m}{2187}\right) \quad \text{if } n \geqq m.$$

From this we can check that $Q_{7,2}(n) > (17/28)n$ for $128 \leqq n \leqq 205$, $256 \leqq n \leqq 616$, $640 \leqq n \leqq 1232$, $1280 \leqq n \leqq 1642$, $1664 \leqq n \leqq 3696$, and $3712 \leqq n \leqq 10854$ by taking successively $m=128$, $m=256$, $m=411$, etc. and using the formula

$$Q_2(u) = \sum_{s=1}^{\infty} \mu(s)[u/s^2],$$

where $\mu(s)$ is the Möbius function of s. We are therefore left to prove (ii) only for $n \leqq 127$, $206 \leqq n \leqq 255$, etc.

If $n \leqq 127$ then $Q_{7,2}(n) = Q_2(n)$, from (2), and (ii) is easily checked. For the remaining values of n, we use the fact that if $n \geqq m$ then

$$Q_{7,2}(n) \geqq Q_2(m) + Q_2(m/128) + Q_2(m/2187)$$

from (2), and check (ii), taking successively $m=206$, $m=210$, $m=213$, etc., thus completing the proof of (ii) for $k=7$.

4. **Proof of (ii) for $k=6$.** From (1), $Q_{6,2}(n) \geqq Q_{7,2}(n)$ and, if $n < 2^6$, $Q_{6,2}(n) = Q_{7,2}(n)$. Hence (ii) for $k=6$ follows from (ii) for $k=7$.

5. **Proof of (iii).** From (1),

$$Q_{5,2}(56) = Q_2(56) + Q_2(1) > Q_{6,2}(56).$$

The rest of the proof is omitted since, apart from the use of this inequality, it is similar to that in §4.

6. **Proof of (iv).** This proof is similar to that in §3 and is omitted.

References

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