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SCHLAIS' THEOREM EXTENDS TO λ CONNECTED PLANE CONTINUA

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ABSTRACT. We call a nondegenerate metric space that is compact and connected a continuum. For each point x of a continuum M, F. B. Jones [2] defines $K(x)$ to be the set consisting of all points y of M such that M is not aposyndetic at x with respect to y. H. E. Schlais [4] proved that if M is a hereditarily decomposable continuum, then for each point x of M, no nonempty open set in M is contained in $K(x)$. A continuum M is said to be λ connected if any two of its points can be joined by a hereditarily decomposable continuum in M. Here we prove that if M is a λ connected plane continuum, then for each point x of M, the set $K(x)$ does not contain a nonempty open subset of M.

A continuum M is said to be *aposyndetic at a point p of M with respect to a point q of M* if there exist an open set U and a continuum H in M such that $p \in U \subset H \subset M - \{q\}$.

For each point x of a continuum M, Jones [2] defines $L(x)$ to be the set consisting of all points y of M such that M is not aposyndetic at y with respect to x. Note that for each point x of M, the set $K(x)$ is closed [2, Theorem 2] and $L(x)$ is connected and closed [2, Theorem 3] in M.

Suppose that M is a plane continuum. In [1] it is proved that the following three statements are equivalent.

I. M is λ connected.

II. For each point x of M, every continuum in $K(x)$ is decomposable.

III. For each point x of M, every continuum in $L(x)$ is decomposable.

Throughout this paper E^2 is the plane. We denote the closure and the boundary of a given set S in E^2 by Cl S and Bd S respectively.

THEOREM. *Suppose that M is a λ connected continuum in E^2. Then for each point x of M, the set $K(x)$ does not contain a nonempty open subset of M.*

PROOF. Assume that for some point x of M, the set $K(x)$ contains a nonempty open subset of M. It follows from [4, Theorem 9, Proof] that

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there exists a sequence $\{U_i\}$ of circular regions with disjoint closures in $E^2-\{x\}$ converging to a point p of $M-\mathrm{Cl}\, U_1$ such that for each positive integer i, (1) $M\cap \mathrm{Cl}\, U_i$ is a nonempty subset of the interior of $K(x)$ (relative to M), and (2) if j is an integer greater than i, then the p-component of $M-U_i$ does not intersect $\mathrm{Cl}\, U_j$.

Let I be a subcontinuum of M that is irreducible with respect to intersecting each element of the sequence $\{\mathrm{Cl}\, U_i\}$. According to [**4**, Theorem 8, Proof], I is indecomposable.

There exists a sequence of disjoint circular regions $\{Z_i\}$ in E^2 that converges to p such that for each positive integer i, the region Z_i contains $\mathrm{Cl}\, U_i$, the set $M\cap Z_i$ is in $K(x)$, and the point x does not belong to $\mathrm{Cl}\, Z_i$.

We now prove that for each positive integer i, every subcontinuum of M that contains a point of I in its interior (relative to M) intersects Z_i. To accomplish this we suppose that there exist a point v of I and a continuum L in M such that v is an interior point of L relative to M and, for some integer i, the set $L\cap Z_i=\varnothing$. There exists a circular region V in E^2-Z_i that meets I such that $V\cap M$ is contained in L and x does not belong to $\mathrm{Cl}\, V$. Since I meets both Z_i and V, it follows from Theorem 1 of [**1**] that there exist continua H and F in I that meet V, arc-segments R and T in V, and a point y of $I\cap Z_i$ such that $H\cup F\cup R\cup T$ separates y from x in E^2. Define D to be the y-component of $E^2-(H\cup F\cup R\cup T)$. There exists a circular region G in E^2 containing y such that $\mathrm{Cl}\, G$ is in $D\cap Z_i$. Define Q to be a circular region in E^2 containing x whose closure misses $H\cup F\cup R\cup T$.

Since $G\cap M$ is a subset of $K(x)$, M is not aposyndetic at x with respect to y. Hence x is not an interior point (relative to M) of the x-component of $M-G$. Since $(H\cup F)\cap G=\varnothing$, there exists a component C of $M-G$ that meets $\mathrm{Bd}\, Q$ such that $(H\cup F)\cap C=\varnothing$. Since $H\cup F\cup R\cup T$ separates $\mathrm{Cl}\, G$ from $\mathrm{Cl}\, Q$ in E^2 and C intersects both $\mathrm{Bd}\, G$ and $\mathrm{Bd}\, Q$, there is a point c of C in $R\cup T$. There exists a simple closed curve J in $(E^2-M)\cup G$ that separates C from H in E^2 [**3**, Theorem 14, p. 171]. Let h be a point of $H\cap V$. The continuum L contains $\{c, h\}$ and does not intersect J. But since J separates c from h in E^2, this is a contradiction. It follows that for each point v of I, every subcontinuum of M that contains v in its interior (relative to M) meets each element of $\{Z_i\}$.

Since the sequence $\{Z_i\}$ converges to p, it follows that $L(p)$ contains I. This contradicts the assumption that M is λ connected [**1**, Theorem 5]. Hence for each point x of M, no nonempty open subset of M is contained in $K(x)$.

Question. Suppose that M is a λ connected (not necessarily planar) continuum. Must the interior of $K(x)$ relative to M be void, for each point x of M?

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