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BEURLING GENERALIZED PRIME NUMBER SYSTEMS IN WHICH THE CHEBYSHEV INEQUALITIES FAIL

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ABSTRACT. It is proved that there exist systems of generalized primes in which the asymptotic distribution of integers is $N(x)=Ax+O(x\cdot\log^{-\gamma}x)$ with $A>0$ and $\gamma\in[0,1)$ and in which the Chebyshev inequalities

$$\liminf_{x\to\infty}\frac{\pi(x)\log x}{x}>0,\qquad \limsup_{x\to\infty}\frac{\pi(x)\log x}{x}<\infty$$

do not hold.

A nondecreasing sequence P of real numbers $p_1, p_2, \cdots$ such that $p_1>1$ and $p_i\to\infty$ is called a Beurling generalized prime number system. The associated system of generalized integers, $N=\{n_i\}_{i=0}^{\infty}$, is the sequence of real numbers obtained by letting $n_0=1$ and arranging in nondecreasing order the elements of the multiplicative semigroup generated by P. The distribution functions $\pi(x)$ and $N(x)$ are defined in the natural way,

$$\pi(x)=\pi_P(x)=\sum_{p_i\leqq x}1,\qquad N(x)=N_P(x)=\sum_{n_i\leqq x}1.$$

Beurling [2] proved that if

$$N(x)=Ax+O(x\log^{-\gamma}x) \tag{1}$$

with $A>0$ and $\gamma>\frac{3}{2}$, then the prime number theorem holds for the system, that is, $\pi(x)\sim x/\log x$. He also showed, essentially, that there exist systems for which (1) holds with $\gamma=\frac{3}{2}$ and in which the theorem is not true. A summary of results and conjectures about systems of generalized primes is given in the recent work of Bateman and Diamond [1].

It is easy to show (as was done by the author in [3]) that if $N(x)=Ax+o(x)$, then

$$\liminf_{x\to\infty}\frac{\pi(x)\log x}{x}\leqq 1,\qquad \limsup_{x\to\infty}\frac{\pi(x)\log x}{x}\geqq 1.$$

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In the following article by Diamond the Chebyshev inequalities

$$\liminf_{x\to\infty} \frac{\pi(x)\log x}{x} \geqq a > 0, \qquad \limsup_{x\to\infty} \frac{\pi(x)\log x}{x} \leqq A < \infty$$

are shown to hold when (1) is satisfied with $\gamma > 1$. In this paper we show that these Chebyshev inequalities need not hold when $\gamma < 1$.

THEOREM. *Let* $\alpha \in [0, 1]$, $\beta \in [1, +\infty]$, *and* $\gamma \in [0, 1)$ *be given. There exists a generalized prime number system in which*

(i) $N(x) = Ax + O(x \log^{-\gamma} x)$,

(ii) $\liminf_{x\to+\infty}(\pi(x)\log x/x) = \alpha$,

(iii) $\limsup_{x\to+\infty}(\pi(x)\log x/x) = \beta$.

PROOF. Let $\pi_0(x)$ be the distribution function for the rational primes. For each rational integer n define the interval $I_n = (a_n, b_n]$ by $\log^{1-\gamma} a_n = 2^n$ and $b_n = a_n 2^{n/4}$. Let C_n be the set consisting of the largest $[(1-\alpha)\{\pi_0(b_n) - \pi_0(a_n)\}]$ rational primes in I_n. If $\beta < +\infty$, let D_n be a set consisting of any $[(\beta-1)\{\pi_0(b_n) - \pi_0(a_n)\}]$ distinct real numbers in the interval $(b_n - 1, b_n]$. If $\beta = +\infty$, let D_n be a set consisting of any $[2^{n/8}\{\pi_0(b_n) - \pi_0(a_n)\}]$ distinct real numbers in $(b_n - 1, b_n]$. Let $C = \bigcup_{\text{odd } n} C_n$, $D = \bigcup_{\text{even } n} D_n$, and let $P = \{p_j\}_{j=1}^{\infty}$ be the nondecreasing sequence formed from the set $P = (R - C) \cup D$, where R is the set of rational primes.

Then for some $K = K(\beta)$ we have

$$\begin{aligned}\sum_{p\in C\cup D} \frac{\log^\gamma p}{p} &\leqq \sum_{\text{odd } n} \{\pi_0(b_n) - \pi_0(a_n)\} \frac{\log^\gamma a_n}{a_n} \\ &\quad + K \sum_{\text{even } n} \frac{2^{n/8}\{\pi_0(b_n) - \pi_0(a_n)\}\log^\gamma a_n}{a_n} \\ &= O\left(\sum_{n=1}^{\infty} \frac{1}{2^{n/2}}\right) = O(1).\end{aligned}$$

Thus we see that $\prod_{p\in C} (1 + \log^\gamma p/p) \cdot \prod_{q\in D} (1 - \log^\gamma q/q)^{-1}$ converges. If for Re $s > 1$ we define $\{c_k\}_{k=1}^{+\infty}$ by

$$\sum_{k=1}^{+\infty} \frac{c_k}{k^s} = \prod_{p\in C} \left(1 - \frac{1}{p^s}\right) \prod_{q\in D} \left(1 - \frac{1}{q^s}\right)^{-1},$$

then we have

$$\sum_{k=1}^{+\infty} \frac{|c_k| \log^\gamma k}{k} \leqq \prod_{p\in C} \left(1 + \frac{\log^\gamma p}{p}\right) \prod_{q\in D} \left(1 - \frac{\log^\gamma q}{q}\right)^{-1} \tag{2}$$

by repeated use of the inequality $\log^\gamma(mn)=(\log m+\log n)^\gamma \leqq \log^\gamma m+\log^\gamma n$.

Define for $\operatorname{Re} s>1$ the function

$$\zeta(s) = \zeta_0(s)\prod_{p\in C}\left(1-\frac{1}{p^s}\right)\prod_{q\in D}\left(1-\frac{1}{q^s}\right)^{-1} = \sum_{n=1}^{+\infty}\frac{a_n}{n^s},$$

where $\zeta_0(s)$ is the Riemann zeta function. Then $\sum_{n=1}^{+\infty} a_n/n^s = (\sum_{m=1}^{+\infty} 1/m^s)\sum_{k=1}^{+\infty} c_k/k^s$ and

$$\begin{aligned} N(x) &= \sum_{n\leqq x} a_n = \sum_{k\leqq x} c_k\left[\frac{x}{k}\right] = x\sum_{k\leqq x}\frac{c_k}{k} + O\left(\sum_{k\leqq x}|c_k|\right) \\ &= x\sum_{k=1}^{+\infty}\frac{c_k}{k} + O\left(x\sum_{k>x}\frac{|c_k|}{k}\right) + O\left(\sum_{k\leqq x}|c_k|\right). \end{aligned}$$

From (2) we have

$$\log^\gamma x\sum_{k>x}\frac{|c_k|}{k} \leqq \sum_{k>x}\frac{|c_k|\log^\gamma k}{k} = O(1)$$

and

$$\sum_{k\leqq x}|c_k| \leqq O(1) + \frac{x}{\log^\gamma x}\sum_{k\leqq x}\frac{|c_k|\log^\gamma k}{k} = O\left(\frac{x}{\log^\gamma x}\right).$$

Thus $N(x)=x\sum_{k=1}^{+\infty} c_k/k+O(x/\log^\gamma x)$.

We next show that (ii) and (iii) hold. Since $\pi_0(x)\sim x/\log x$ we have

$$\lim_{n\to\infty}\frac{\pi_0(a_n)}{\pi_0(b_n)} = 0 \quad\text{and}\quad \lim_{n\to\infty}\frac{\pi_0(b_{n-1})}{\pi_0(a_n)}2^{n/8} = 0.$$

From the fact that

$$\begin{aligned} \pi(x) &\geqq \pi_0(x) - \sum_{b_n\leqq x;\, n \text{ odd}}[(1-\alpha)\{\pi_0(b_n)-\pi_0(a_n)\}] \\ &\geqq \pi_0(x) - (1-\alpha)\pi_0(x) \end{aligned}$$

it is clear that

$$\liminf_{x\to+\infty}\frac{\pi(x)\log x}{x} = \liminf_{x\to+\infty}\frac{\pi(x)}{\pi_0(x)} \geqq \alpha.$$

On the other hand

$$\begin{aligned} \pi(a_n) &\leqq \pi_0(a_n) + K\sum_{k<n;\, k \text{ even}} 2^{k/8}\{\pi_0(b_k)-\pi_0(a_k)\} \\ &\leqq \pi_0(a_n) + K2^{n/8}\pi_0(b_{n-1}). \end{aligned}$$

It follows then that $\lim_{n\to\infty} \pi(a_n)/\pi_0(b_n)=0$ and we have (i), since

$$\liminf_{x\to\infty} \frac{\pi(x)}{\pi_0(x)} \leqq \liminf_{n\to\infty;n \text{ odd}} \frac{\pi(b_n)}{\pi_0(b_n)}$$
$$= \liminf_{n\to\infty;n \text{ odd}} \frac{\pi(a_n) + \alpha\{\pi_0(b_n) - \pi_0(a_n)\}}{\pi_0(b_n)} = \alpha.$$

If $\beta<\infty$, then

$$\pi(x) \leqq \pi_0(x) + \sum_{b_n \leqq x;n \text{ even}} (\beta - 1)\{\pi_0(b_n) - \pi_0(a_n)\} \leqq \beta\pi_0(x).$$

Similarly, it follows that

$$\limsup_{x\to\infty} \frac{\pi(x)}{\pi_0(x)} \geqq \limsup_{n\to\infty;n \text{ even}} \frac{\pi(b_n)}{\pi_0(b_n)}$$
$$= \limsup_{n\to\infty;n \text{ even}} \frac{\pi(a_n) + \beta\{\pi_0(b_n) - \pi_0(a_n)\}}{\pi_0(b_n)} = \beta.$$

Thus, $\limsup_{x\to\infty} \pi(x)\log x/x=\beta$. If $\beta=\infty$, then

$$\limsup_{x\to\infty} \frac{\pi(x)}{\pi_0(x)} \geqq \limsup_{n\to\infty} \frac{\pi(b_n)}{\pi_0(b_n)}$$
$$= \limsup_{n\to\infty} \frac{\pi(a_n) + (2^{n/8} + 1)\{\pi_0(b_n) - \pi_0(a_n)\}}{\pi_0(b_n)} = \infty.$$

This completes the proof.

A slightly more complicated argument shows that (ii) and (iii) can hold for systems in which $N(x)=Ax+O(x/g(x))$ provided $g(x)=o(\log x)$. Details may be found in [3].

References

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