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PRIME RINGS WITH INVOLUTION WHOSE SYMMETRIC ZERO-DIVISORS ARE NILPOTENT

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ABSTRACT. Let k be a field and R the k-algebra generated by x and y with the single defining relation $x^2=0$. Using free ring techniques we prove that the set of left zero-divisors of R is Rx. There is a unique involution fixing x, y and this makes R into a prime ring with involution whose symmetric zero-divisors are nilpotent (answering a question by W. S. Martindale). This example also provides us with a subfunctor of the identity whose value is a one-sided ideal (answering a question by R. Baer).

Recently W. S. Martindale (in conversation) raised the question whether a prime ring with involution exists, not 4-dimensional over its centre, in which every symmetric zero-divisor is nilpotent. He observed that there is a natural candidate over any field k, namely the algebra $k\langle x, y|x^2=0\rangle$, with involution over k fixing x and y, but to verify the desired property seems not entirely trivial. The proof given here[1] illustrates the use of free ring techniques described in [1]. In fact we shall prove the

THEOREM. *Let R be an algebra over a field k, generated by x, y with the single defining relation $x^2=0$; then any left zero-divisor is in Rx.*

PROOF. Let $F=k\langle x, y\rangle$ be the free k-algebra on x and y, and let $\mathfrak{a}$ be the ideal of F generated by x^2; then the monomials which do not have x^2 as factor are k-linearly independent mod $\mathfrak{a}$. Given a, $b\in F$ such that a, $b\notin\mathfrak{a}$ but $ab\in\mathfrak{a}$, it follows that neither a nor b has a constant term. Write $a=a_0+a_1$, $b=b_0+b_1$, where a_0, a_1 end in x, y respectively and b_0, b_1 begin with x, y respectively. By omitting terms we may assume that neither a nor b contains a monomial with x^2 as factor, and since the others are linearly independent mod $\mathfrak{a}$, we have $a_0b_1+a_1b_0+a_1b_1=0$, i.e.

$$ab = a_0b_0 \quad \text{in } F. \tag{1}$$

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[1] Professor Martindale informs me that Estes has also obtained a proof. Another proof, using coproducts, has been found by G. M. Bergman.

This equation shows that $a_1=0$ if and only if $b_1=0$, i.e. $a \in Fx$ iff $b \in xF$. To complete the proof we use induction on the number of factors of a. If a is an atom (i.e. unfactorable) then since a, a_0 have no constant term, their highest common left factor is a nonunit, and is therefore a, so $a_0=ac$. By comparing degrees we see that c is a unit, so $a=c^{-1}a_0 \in Fx$ as claimed. Next let $a=a'p$, where p is an atom, then $p \notin \mathfrak{a}$; if $pb \in \mathfrak{a}$, $p \in Fx$ and so $a \in Fx$. Hence we may assume that $pb \notin \mathfrak{a}$, but $a'pb=ab \in \mathfrak{a}$, so by the induction hypothesis, $a'=a''x$, $pb=xc$, and it follows by the weak algorithm [**1**, Chapter 2] that $p=xq+\lambda$, where $\lambda \in k$. If $\lambda=0$, p is associated to x and the result follows, so assume $\lambda \neq 0$. Then $xc=(xq+\lambda)b$, hence $\lambda b=x(c-qb)$, so $b \in xF$, and therefore $a \in Fx$, as we wished to show.

From the theorem it is clear that R is prime: if a, $b \neq 0$, then $ayb \neq 0$, so $aRb \neq 0$, and of course R is infinite dimensional and central over k. Moreover, we have the

COROLLARY. *If in R we take the involution fixing x, y, then every symmetric zero-divisor is nilpotent.*

For if c is a symmetric zero-divisor, say $cd=0$, then $c=ax$, and by symmetry, $c=xa^*$, hence $c^2=ax^2a^*=0$.

The above theorem can also be used to solve the following problem, raised by R. Baer: Find a ring R with a left ideal invariant under all automorphisms of R, yet not two-sided. The ring R of the theorem and Rx clearly form such a pair. More generally, Baer asks for a subfunctor of the identity whose value is a left ideal which is not always two-sided. The first examples that come to mind are one-sided functors (i.e. not invariant under passage to the opposite ring), but their values are usually two-sided ideals, e.g. the left socle. An example meeting the above requirements is "the left ideal generated by all the nilpotent elements". This is easily verified to be a subfunctor of the identity and, in the ring R of the theorem, its value is Rx.

REFERENCE

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