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TWO ERGODIC THEOREMS FOR CONVEX COMBINATIONS OF COMMUTING ISOMETRIES[1]

S. A. McGRATH

ABSTRACT. Let $(X, \mathscr{F}, \mu)$ be a measure space. In this paper we obtain L^p estimates for the supremum of the Cesàro averages of combinations of commuting isometries of $L^p(X, \mathscr{F}, \mu)$. In particular, we show that a convex combination of two invertible commuting isometries of $L^p(X, \mathscr{F}, \mu)$, p fixed, $1<p<\infty$, $p\neq 2$, admits of a dominated estimate with constant $p/(p-1)$. We also show that a convex combination of an arbitrary number of commuting positive invertible isometries of $L^2(X, \mathscr{F}, \mu)$ admits of a dominated estimate with constant 2.

1. **Introduction.** Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space and let T be a linear operator mapping $L^p(X, \mathscr{F}, \mu)$ into $L^p(X, \mathscr{F}, \mu)$, p fixed, $1<p<\infty$. If there exists a constant $c>0$ such that

$$\int \sup_n \left| f, \frac{f+Tf}{2}, \cdots, \frac{f+Tf+\cdots+T^{n-1}f}{n} \right|^p d\mu \leqq c^p \int |f|^p \, d\mu$$

for all $f \in L^p(X, \mathscr{F}, \mu)$, then we say that T admits of a dominated estimate with constant c. If the norm of T does not exceed 1, then we say that T is a contraction. We say that T is positive if it maps nonnegative functions to nonnegative functions.

The purpose of this paper is to obtain dominated estimates for certain convex combinations of commuting invertible isometries of $L^p(X, \mathscr{F}, \mu)$. In [5, pp. 368–371], A. Ionescu Tulcea showed that if T is an invertible isometry of $L^p(X, \mathscr{F}, \mu)$, $p\neq 2$, then it admits of a dominated estimate with constant $p/(p-1)$. For the case $p=2$, she showed that a positive invertible isometry admits of a dominated estimate with constant 2. We extend these results by showing that if $p\neq 2$, a convex combination of two commuting invertible isometries of $L^p(X, \mathscr{F}, \mu)$ admits of a dominated estimate with

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constant $p/(p-1)$. In the case $p=2$, we show that a convex combination of an arbitrary number of commuting positive invertible isometries admits of a dominated estimate with constant 2 by obtaining this estimate for normal positive contractions. This last result generalizes a result due to E. M. Stein, who showed in [**4**, Theorem 2, p. 1896] that positive selfadjoint contractions admit of dominated estimates with constant 4.

2. **Preliminary theorems.** All estimates are obtained by applying the following results.

LEMMA 2.1. *If T_n $(n=1, 2, \cdots)$ and T are contractions of $L^p(X, \mathscr{F}, \mu)$ such that T_n converges strongly to T, and such that each T_n admits of a dominated estimate with constant c, then T admits of a dominated estimate with constant c.*

THEOREM 2.2. *Let $(X_i, \mathscr{F}_i, \mu_i)$, $i=1, 2, \cdots, n$, be σ-finite measure spaces and let T_i be a positive contraction of $L^p(X_i, \mathscr{F}_i, \mu_i)$ into $L^p(X_{i+1}, \mathscr{F}_{i+1}, \mu_{i+1})$ for $i=1, 2, \cdots, n-1$. Let $h_i \in L^p(X_i, \mathscr{F}_i, \mu_i)$ be a positive function, $i=1, 2, \cdots, n$, such that* (i) $T_i h_i \leqq h_{i+1}$, (ii) $T_i^*(h_{i+1}^{p-1}) \leqq h_i^{p-1}$, *for $i=1, 2, \cdots, n-1$. Then we have*

$$\sum_{i=1}^{n} \int \sup \left| f_i, \frac{f_i + T_{i-1} f_{i-1}}{2}, \cdots, \frac{f_i + T_{i-1} f_{i-1} + \cdots + T_{i-1} T_{i-2} \cdots T_1 f_1}{i} \right|^p d\mu_i$$
$$\leqq \left(\frac{p}{p-1} \right)^p \sum_{i=1}^{n} \int |f_i|^p \, d\mu_i$$

where $f_i \in L^p(X_i, \mathscr{F}_i, \mu_i)$.

A slightly more general statement of this theorem and its proof can be found in [**1**, Theorem 2.1, p. 612].

COROLLARY 2.3. *Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space and let T be a positive contraction of $L^p(X, \mathscr{F}, \mu)$. Suppose there exists a positive function $h \in L^p(X, \mathscr{F}, \mu)$ such that either* (i) $Th=h$, *or* (ii) $Th \leqq h$ *and* $T^*(h^{p-1}) \leqq h^{p-1}$. *Then T admits of a dominated estimate with constant $p/(p-1)$.*

PROOF. If h satisfies condition (ii) then the corollary follows from Theorem 2.2 upon setting $(X_i, \mathscr{F}_i, \mu_i)=(X, \mathscr{F}, \mu)$, $T_i=T$, $h_i=h$, and $f_i = f$ for $i=1, 2, \cdots, n$ and noting that

$$\int \sup_n \left| f, \frac{f + Tf}{2}, \cdots, \frac{f + Tf + \cdots + T^{n-1} f}{n} \right|^p d\mu$$

equals the limit of the Cesàro sums. It is shown in [**2**, Lemma 3.1, p. 267] that if h is a positive fixed function, i.e. h satisfies condition (i), then $T^*(h^{p-1}) = h^{p-1}$. Thus if $Th=h$, then T admits of a dominated estimate with constant $p/(p-1)$. Q.E.D.

3. **Main results.** We obtain the estimate for normal contractions first.

THEOREM 3.1. *Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space and let T be a positive normal contraction of $L^2(X, \mathscr{F}, \mu)$. Then T admits of a dominated estimate with constant* 2.

PROOF. Suppose first that $\|T\|<1$. Then $\|T^*\|<1$. Let f be any positive function in $L^2(X, \mathscr{F}, \mu)$ and let $g=\sum_{k=0}^{\infty} T^{*k}(f)$. Then $g>0$, $g \in L^2(X, \mathscr{F}, \mu)$, and $T^*g \leqq g$. Let $h=\sum_{k=0}^{\infty} T^k g$. We have $h>0$, $h \in L^2(X, \mathscr{F}, \mu)$, and $Th \leqq h$. Moreover, $T^*h \leqq h$ since $T^*h=\sum_{k=0}^{\infty} T^k(T^*g) \leqq \sum_{k=0}^{\infty} T^k g=h$. Thus T admits of a dominated estimate with constant 2 by Corollary 2.3.

Now suppose $\|T\| \leqq 1$. Define $T_n=(1-2^{-n})T$, $n=1, 2, 3, \cdots$. Then each T_n admits of a dominated estimate with constant 2 and consequently so does T since T_n converges strongly to T. Q.E.D.

COROLLARY 3.2. *Let T_i, $i=1, 2, \cdots$, be positive invertible isometries of $L^2(X, \mathscr{F}, \mu)$ which commute, i.e. $T_i T_j=T_j T_i$ for all i, j. Then if $\alpha_i \geqq 0$, $i=1, 2, \cdots$ and $\sum_{i=1}^{\infty} \alpha_i \leqq 1$, the convex combination $S=\sum_{i=1}^{\infty} \alpha_i T_i$ admits of a dominated estimate with constant* 2.

PROOF. Since $T_i^*=T_i^{-1}$, each T_i is normal and each T_i commutes with T_j^*, $j=1, 2, \cdots$. Consequently $S_n=\sum_{i=1}^{n} \alpha_i T_i$ is a positive normal contraction which admits of a dominated estimate by Theorem 3.1. Since $S_n \rightarrow S$ strongly, S also admits of a dominated estimate with constant 2 by Lemma 2.1. Q.E.D.

We now consider the case of convex combinations of isometries of $L^p(X, \mathscr{F}, \mu)$, $p \neq 2$. To avoid technical difficulties we will assume that $(X, \mathscr{F}, \mu)$ is a Lebesgue space although the results are true in general.

LEMMA 3.3. *Let $(X, \mathscr{F}, \mu)$ be a Lebesgue space and A, B invertible isometries of $L^p(X, \mathscr{F}, \mu)$, $p \neq 2$. Let $0<\alpha<1$ and $\beta=1-\alpha$. Then given $n>0$ and $f \in L^p(X, \mathscr{F}, \mu)$, we have*

$$\frac{1}{n} \sum_{i=0}^{n-1} \int \sup \left| A^i f, \frac{A^i f+SA(A^i f)}{2}, \cdots, \frac{A^i f+SA(A^i f)+\cdots+S^{n-i-1} A^{n-i-1}(A^i f)}{n-i} \right|^p d\mu \leqq \left(\frac{p}{p-1}\right)^p \int |f|^p \, d\mu,$$

where $S=\alpha+\beta BA^{-1}$.

PROOF. It is a result essentially due to Banach (see [**3**, Theorem 3.1, p. 461] for a proof of a generalized version of Banach's theorem) that an invertible isometry U can be represented in the form $Uf(x)=f(\tau x) \cdot r(x)$, where τ is an invertible measurable point transformation and r is a measurable function satisfying $|r|^p=d(\tau\mu)/d\mu$. Since BA^{-1} is an invertible isometry, it can be written in the form $BA^{-1}(f(x))=f(\tau x)\cdot r(x)$. Defining $Uf(x)=f(\tau x)|r(x)|$, we see that U is a positive invertible isometry which dominates BA^{-1} in the sense that for $n=1, 2, 3, \cdots$

$$U^n(|f(x)|) \geqq |(BA^{-1})^n f(x)|$$

for any $f\in L^p(X, \mathscr{F}, \mu)$. Let $\{U_m\}$ be a sequence of positive periodic isometries which converges strongly to U. A proof that such a sequence exists can be found in [**2**, Lemma 4.3, p. 269]. A positive periodic isometry T of period n (T has period n if $T^n=I$) has a positive fixed function since if $h>0$ then $h+Th+\cdots+T^{n-1}h$ is a positive fixed function of T. Thus each U_m has a positive fixed function g_m and consequently so does each operator $S_m=\alpha+\beta U_m$ since $\alpha+\beta=1$. Letting $T_i=S_m$, for $i=1, 2, \cdots, n-1$ in Theorem 2.2, we have

$$\sum_{i=1}^{n}\int \sup\left|f_i, \frac{f_i+S_m f_{i-1}}{2}, \cdots, \frac{f_i+S_m f_{i-1}+\cdots+S_m^{i-1}f_1}{i}\right|^p d\mu$$
$$\leqq \left(\frac{p}{p-1}\right)^p \sum_{i=1}^{n}\int |f_i|^p\, d\mu,$$

where $f_i\in L^p(X, \mathscr{F}, \mu)$, $i=1, 2, \cdots, n$. Since this inequality holds for each S_m, it follows that it holds with $D=\alpha+\beta U$ in place of S_m since S_m converges to D. Since D dominates S, we have

$$\sum_{i=1}^{n}\int \sup\left|f_i, \frac{f_i+Sf_{i-1}}{2}, \cdots, \frac{f_i+Sf_{i-1}+\cdots+S^{i-1}f_1}{i}\right|^p d\mu$$
$$\leqq \left(\frac{p}{p-1}\right)^p \sum_{i=1}^{n}\int |f_i|^p\, d\mu$$

for $f_i\in L^p(X, \mathscr{F}, \mu)$, $i=1, 2, \cdots, n$.

We let $f_i=A^{n-i}f$, $i=1, 2, \cdots, n$ in the preceding inequality and reverse the order of summation, getting

$$\frac{1}{n}\sum_{i=0}^{n-1}\int \sup\left|A^i f, \frac{A^i f+SA(A^i f)}{2}, \cdots,\right.$$
$$\left.\frac{A^i f+SA(A^i f)+\cdots+S^{n-i-1}A^{n-i-1}(A^i f)}{n-i}\right|^p d\mu$$
$$\leqq \left(\frac{p}{p-1}\right)^p \int |f|^p\, d\mu. \qquad \text{Q.E.D.}$$

THEOREM 3.4. *Let $(X, \mathscr{F}, \mu)$ be a Lebesgue space and A, B commuting invertible isometries of $L^p(X, \mathscr{F}, \mu)$, $p\neq 2$. Let $0<\alpha<1$ and $\beta=1-\alpha$. Then $\alpha A+\beta B$ admits of a dominated estimate with constant $p/(p-1)$.*

PROOF. Defining $S=\alpha+\beta BA^{-1}$ we have by the preceding lemma that

$$\frac{1}{n}\sum_{i=0}^{n-1}\int \sup\left|A^if, \frac{A^if+SA(A^if)}{2}, \cdots, \frac{A^if+\cdots+S^{n-i-1}A^{n-i-1}(A^if)}{n-i}\right|^p d\mu \leqq \left(\frac{p}{p-1}\right)^p\int |f|^p\, d\mu$$

for any positive integer n and any $f\in L^p(X, \mathscr{F}, \mu)$. Since A and B commute we have

$$\frac{1}{n}\sum_{i=0}^{n-1}\int \sup\left|A^if, \frac{A^if+C(A^if)}{2}, \cdots, \frac{A^if+C(A^if)+\cdots+C^{n-i-1}(A^if)}{n-i}\right|^p d\mu \leqq \left(\frac{p}{p-1}\right)^p\int |f|^p\, d\mu,$$

where $C=\alpha A+\beta B$. Since A is an isometry it can be written in the form $Ag(x)=g(\tau x)r(x)$. It thus follows that

$$\begin{aligned}\int \sup&\left|A^if, \frac{A^if+C(A^if)}{2}, \cdots, \frac{A^if+C(A^if)+\cdots+C^{n-i-1}(A^if)}{n-i}\right|^p d\mu\\ &=\int \sup\left|A^if, \frac{A^if+A^i(Cf)}{2}, \cdots, \frac{A^if+A^i(Cf)+\cdots+A^i(C^{n-i-1}f)}{n-i}\right|^p d\mu\\ &=\int \left|A^i\left(\sup\left|f, \frac{f+Cf}{2}, \cdots, \frac{f+Cf+\cdots+C^{n-i-1}f}{n-i}\right|\right)\right|^p d\mu\\ &=\int \sup\left|f, \frac{f+Cf}{2}, \cdots, \frac{f+Cf+\cdots+C^{n-i-1}f}{n-i}\right|^p d\mu\end{aligned}$$

for $i=0, 1, \cdots, n-1$. Thus for any n, we have

$$\frac{1}{n}\sum_{i=0}^{n-1}\int \sup\left|f, \frac{f+Cf}{2}, \cdots, \frac{f+Cf+\cdots+C^{n-i-1}(f)}{n-i}\right|^p d\mu \leqq \left(\frac{p}{p-1}\right)^p\int |f|^p\, d\mu \quad \text{for any } f\in L^p(X, \mathscr{F}, \mu).$$

Since the Cesàro sums do not exceed $(p/(p-1))^p\int |f|^p\,d\mu$, it follows that

$$\int \sup \left| f, \frac{f + Cf}{2}, \cdots, \frac{f + Cf + \cdots + C^{n-1}f}{n} \right|^p d\mu$$
$$\leqq \left(\frac{p}{p-1}\right)^p \int |f|^p\,d\mu \quad \text{Q.E.D.}$$

It has been pointed out by M. A. Ackoglu that for contractions, the existence of a dominated estimate implies the almost everywhere existence of the Cesàro limit.

Hence we have:

THEOREM 3.5. *Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space. Then the individual ergodic theorem holds for positive normal contractions of $L^2(X, \mathscr{F}, \mu)$ and for convex combinations of 2 invertible commuting isometries of $L^p(X, \mathscr{F}, \mu)$, $p\neq 2$.*

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U.S. NAVAL ACADEMY, ANNAPOLIS, MARYLAND 21402