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INVERSION OF NORMAL OPERATORS BY POLYNOMIAL INTERPOLATION[1]

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ABSTRACT. The inverse of a bounded normal linear operator on a Hilbert space is uniformly approximated by a sequence of Newton interpolation polynomials, provided the operator's spectrum lies in either the right or left open half-planes.

1. **Introduction.** The following result is established in **[1]**:

THEOREM 1. *Let $\mathscr{H}$ be a complex Hilbert space, and let $A:\mathscr{H}\to\mathscr{H}$ be bounded, linear, and normal, with spectrum in the open right half-plane. Furthermore, let $\{\omega_N\}$: natural numbers $\to(0, 1]$ satisfy the conditions*

$$\lim_{N\to\infty} \omega_N = 0, \qquad \sum_{N=1}^{\infty} \omega_N = \infty. \tag{1}$$

Then the corresponding recursive averaging process,

$$\Phi_{N+1} = (1-\omega_N)\Phi_N + \omega_N[(I-A)\Phi_N + I], \qquad \Phi_1 = I, \tag{2}$$

generates a sequence of bounded linear operators Φ_N which converge uniformly to A^{-1}.

In a certain sense, this result is an extension of the classical Neumann series representation of A^{-1}, e.g., when $\omega_N=1$, $\forall N$, (2) generates the partial sums of the Neumann series, which converges to A^{-1} iff $\|I-A\|=\sup_{\|x\|=1}\|(I-A)x\|<1$. Here, we establish another relationship between Theorem 1 and classical numerical analysis; specifically, we will prove that when $\omega_N=1/(N+1)$, the corresponding polynomials in A generated by (2) are formally identical to Newton interpolation polynomials for the real function, $f(x)=1/x$.

2. **Results.** Let x_0, $h \in R^1$ and let $x_i=x_0+ih$, $0\leqq i\leqq N$; then the corresponding Nth degree forward Newton interpolation polynomial for $f: R^1\to R^1$ is given by:

$$P_N(x) = f(x_0) + \sum_{i=0}^{N-1} \frac{\Delta^{i+1} f_0}{(i+1)!}\left\{\prod_{k=0}^{i}\left(\frac{x-x_0}{h} - k\right)\right\}$$

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where $\Delta^i f_0 = \sum_{j=0}^{i} ((-1)^j i!/j!(i-j)!) f(x_{i-j})$ (cf. [2]). Put $f(x)=1/x$, $h=1$, and $x_i = i+1$, $0 \leqq i \leqq N$; then

$$\Delta^i f_0 = \sum_{j=0}^{i} \frac{(-1)^j i!}{j!\,(i+1-j)!} = \frac{1}{i+1} \sum_{j=0}^{i+1} \frac{(-1)^j (i+1)!}{j!\,(i+1-j)!} - \frac{(-1)^{i+1} i!}{(i+1)!\,0!}$$
$$= \frac{1}{i+1} (\xi - 1)^{i+1} \Big|_{\xi=1} + \frac{(-1)^i}{i+1} = \frac{(-1)^i}{i+1}$$

and consequently,

$$P_N(x) = 1 + \sum_{i=0}^{N-1} \frac{1}{(i+2)!} \left\{ \prod_{k=0}^{i} (k+1-x) \right\}. \tag{3}$$

We will now prove that (3) is formally identical to the polynomials generated by (2) when

$$\omega_N = 1/(N+1). \tag{4}$$

Thus, the process (2) corresponding to (4) inverts A by interpolating the operator function, $f(A) = A^{-1}$ between integral multiples of the identity operator, I.

LEMMA 1.

$$x\left(1 + \sum_{i=0}^{N-1} \frac{1}{(i+2)!} \left\{ \prod_{k=0}^{i} (k+1-x) \right\}\right)$$
$$= 1 - \frac{1}{(N+1)!} \left\{ \prod_{k=0}^{N} (k+1-x) \right\}.$$

PROOF. By induction on N. For $N=1$, we have $x(1+(1/2!)\{(1-x)\}) = 1-(1/2!)\{(1-x)(2-x)\}$. If the lemma is true for $N=M$, we then have, for $N=M+1$,

$$x\left(1 + \sum_{i=0}^{M} \frac{1}{(i+2)!} \left\{ \prod_{k=0}^{i} (k+1-x) \right\}\right)$$
$$= x\left(1 + \sum_{i=0}^{M-1} \frac{1}{(i+2)!} \left\{ \prod_{k=0}^{i} (k+1-x) \right\}\right) + \frac{1}{(M+2)!} \left\{ \prod_{k=0}^{M} (k+1-x) \right\}$$
$$= 1 + \left(\frac{x}{(M+2)!} - \frac{1}{(M+1)!} \right) \left\{ \prod_{k=0}^{M} (k+1-x) \right\}$$
$$= 1 - \frac{1}{(M+2)!} \left\{ \prod_{k=0}^{M+1} (k+1-x) \right\}.$$

Consequently, the lemma holds also for $N=M+1$, and therefore for all N. Q.E.D.

LEMMA 2.

$$1+\sum_{i=0}^{N-1}\frac{1}{(i+2)!}\left\{\prod_{k=0}^{i}(k+1-x)\right\} = \frac{1}{N+1}\left(1+\sum_{i=1}^{N}\left\{\prod_{k=i}^{N}\frac{(k+1-x)}{k}\right\}\right).$$

PROOF. By induction on N. For $N=1$, we have $1+(1/2!)\{(1-x)\}=(1/2)(1+\{(2-x)/1\})$. If the lemma is true for $N=M$, we then have for $N=M+1$,

$$\begin{aligned}
&\frac{1}{M+2}\left(1+\sum_{i=1}^{M+1}\left\{\prod_{k=i}^{M+1}\frac{(k+1-x)}{k}\right\}\right)\\
(5)\quad &= \frac{1}{M+2}\left(1+\frac{(M+2-x)}{M+1}\left[1+\sum_{i=1}^{M}\left\{\prod_{k=i}^{M}\frac{(k+1-x)}{k}\right\}\right]\right)\\
&= \frac{1}{M+2}\left(1+(M+2-x)\left[1+\sum_{i=0}^{M+1}\frac{1}{(i+2)!}\left\{\prod_{k=0}^{i}(k+1-x)\right\}\right]\right).
\end{aligned}$$

By applying Lemma 1 to the right side of (5) and simplifying, we obtain

$$\frac{1}{M+2}\left(1+\sum_{i=1}^{M+1}\left\{\prod_{k=i}^{M+1}\frac{(k+1-x)}{k}\right\}\right) = 1+\sum_{i=0}^{M}\frac{1}{(i+2)!}\left\{\prod_{k=0}^{i}(k+1-x)\right\};$$

hence the lemma holds also for $N=M+1$, and therefore for all N. Q.E.D.

THEOREM 2. *Let $A:\mathscr{H}\to\mathscr{H}$ be a linear operator and let Φ_N denote the corresponding sequence of linear operators generated by the recursive averaging process,*

$$\Phi_{N+1} = \frac{N}{N+1}\Phi_N + \frac{1}{N+1}[(I-A)\Phi_N + I], \qquad \Phi_1 = I.$$

Then for $N\geqq 1$, $\Phi_{N+1}=P_N(A)$.

PROOF. By induction on N, and Lemma 2. For $N=1$, we have

$$\Phi_2 = (1/2)I + (1/2)[(I-A)I+I] = I + (1/2!)(I-A) = P_1(A).$$

If the theorem is true for $N=M$, we then have for $N=M+1$,

$$\begin{aligned}\Phi_{M+2} &= \frac{1}{M+2}([(M+2)I - A]P_N(A) + I)\\ &= \frac{1}{M+2}\left(I + \frac{[(M+2)I-A]}{M+1}\left[I + \sum_{i=1}^{M}\left\{\prod_{k=i}^{M}\frac{[(k+1)I-A]}{k}\right\}\right]\right)\\ &= \frac{1}{M+2}\left(I + \sum_{k=1}^{M+1}\left\{\prod_{k=i}^{M+1}\frac{[(k+1)I-A]}{k}\right\}\right) = P_{N+1}(A)\end{aligned}$$

by Lemma 2. Q.E.D.

3. **Conclusions.** In view of Theorems 1 and 2, we conclude that the forward Newton interpolation polynomials $P_N(A)$ converge uniformly to A^{-1} if A is a bounded, normal, linear operator with spectrum in the open right half-plane. An analogous conclusion holds if the spectrum of A falls in the open left half-plane, viz., the operators, Ψ_N generated by

$$\Psi_{N+1} = \frac{N}{N+1}\Psi_N + \frac{1}{N+1}[(I+A)\Psi_N - I], \qquad \Psi_1 = I,$$

converge uniformly to A^{-1}, and are formally identical to backward Newton interpolation polynomials $Q_N(x)$ for $f(x)=1/x$ (on negative integral multiples of the identity). Furthermore, it follows from a second result of [**1**] that for all $y \in$ range of A, $P_N(A)y$ converges at least weakly to some solution of $Ax=y$, provided A is a bounded normal linear operator with spectrum in any member of the family of closed disks,

$$\text{(6)}\quad \mathscr{D}(\omega) = \{\zeta \in \mathscr{C}^1 \mid (\mathrm{Re}(\zeta) - 1/\omega)^2 + (\mathrm{Im}(\zeta))^2 \leqq (1/\omega)^2\}, \quad \omega \in (0, 1].$$

Once again, an analogous result obtains if the spectrum of A falls in the reflection of (6) in the imaginary axis.

References

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