PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 40, Number 1, September 1973

# TYPE II $W$-* ALGEBRAS ARE NOT NORMAL

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ABSTRACT. If $\mathscr{A}$ is a $W$-* algebra of type II on separable Hilbert space $H$, then $\mathscr{A}$ is not normal.

Let $\mathscr{A}$ be a $W$-* algebra on separable Hilbert space $H$, and let $\mathscr{Z}$ be the center of $\mathscr{A}$. A $W$-* subalgebra $\mathscr{B}$ of $\mathscr{A}$ is *full* in $\mathscr{A}$ if $\mathscr{Z} \subset \mathscr{B} \cap \mathscr{B}'$. We define the *relative commutant* $\mathscr{B}^c$ of $\mathscr{B}$ in $\mathscr{A}$ to be $\mathscr{B}^c = \mathscr{B}' \cap \mathscr{A}$, and say that $\mathscr{B}$ is *normal* in $\mathscr{A}$ if $\mathscr{B} = \mathscr{B}^{cc}$. Clearly if $\mathscr{B}$ is normal then $\mathscr{B}$ is full. We say that $\mathscr{A}$ is *normal* if $\mathscr{B}$ is normal in $\mathscr{A}$ for every full subalgebra $\mathscr{B}$ of $\mathscr{A}$.

Every type I factor is normal [**3**, Lemma 11.2.2], and no type II factor is normal [**2**, Theorem 3]. Although nonnormal type III factors exist [**5**, Lemma 4.4.2(iii)], no general theorem has been proved concerning this case.

As for results on normality for general $W$-* algebras, it is well known that any type I $W$-* algebra is normal [**1**, p. 307, Exercise 13]. In this paper we apply direct integral theory to show that if $\mathscr{A}$ is of type II then $\mathscr{A}$ is not normal.

Let $\mathscr{A} = \int_\Lambda \oplus \mathscr{A}(\lambda)\mu(d\lambda)$ be the direct integral decomposition of $\mathscr{A}$ into factors. For general information on direct integrals see [**4**], [**6**]. In particular, $K$ denotes the underlying separable Hilbert space of $H$. For $\mathscr{A}$ a finite $W$-* algebra, $|T|_2$ denotes the trace norm of $T \in \mathscr{A}$ [**6**, Definition 2.6]. $\mathscr{S}$ denotes the unit ball in $\mathscr{B}(K)$ taken with the strong-* operator topology.

If $\mathscr{B}$ is full, then $\mathscr{B} = \int_\Lambda \oplus \mathscr{B}(\lambda)\mu(d\lambda)$ with $\mathscr{B}(\lambda) \subset \mathscr{A}(\lambda)$ $\mu$-a.e. $\lambda$. The following simple lemma is a key to our argument.

LEMMA 1. *$\mathscr{B} = \mathscr{B}^{cc}$ if and only if $\mathscr{B}(\lambda) = \mathscr{B}(\lambda)^{cc}$* $\mu$-a.e.

PROOF. $\mathscr{B}^{cc} = \int_\Lambda \oplus \mathscr{B}(\lambda)^{cc}\mu(d\lambda)$. Q.E.D.

The next corollary is obvious.

COROLLARY 2. *$\mathscr{B}$ is normal in $\mathscr{A}$ if and only if $\mathscr{B}$ is full and $\mathscr{B}(\lambda)$ is normal in $\mathscr{A}(\lambda)$* $\mu$-a.e.

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Received by the editors November 9, 1972.

*AMS (MOS) subject classifications* (1970). Primary 46L10.

*Key words and phrases.* $W$-* algebra, normal, hyperfinite, direct integral decomposition.

COROLLARY 3. *If $\mathscr{A}$ is a type* I *$W$-* algebra, $\mathscr{A}$ is normal.*

PROOF. If $\mathscr{A}$ is type I, $\mathscr{A}(\lambda)$ is type I $\mu$-a.e. and hence normal. Q.E.D.

The argument of Corollary 3 clearly shows that if $\mathscr{A}(\lambda)$ is normal $\mu$-a.e. then $\mathscr{A}$ is normal. If the converse could be proved, we would achieve our goal. The author has not been able to prove this, however. Instead, we shall adapt the arguments of [**2**] concerning hyperfinite factors to our more general setting.

A $W$-* algebra $\mathscr{A}$ is *hyperfinite* if it is generated by the union of an increasing sequence of finite dimensional $W$-* subalgebras. Let $\mathscr{A}$ be of type $\mathrm{II}_1$. To show that $\mathscr{A}$ of type $\mathrm{II}_1$ is nonnormal, it suffices to show that, if $\mathscr{F}$ denotes the hyperfinite $\mathrm{II}_1$ factor (by [**4**, Theorem II.6.9] all hyperfinite $\mathrm{II}_1$ factors are isomorphic), then, for $\mathscr{A}(\lambda)\cong\mathscr{F}$ either for all $\lambda$ or for no $\lambda$, $\mathscr{A}$ is nonnormal. Indeed, this follows from the fact that the set $\{\lambda|\mathscr{A}(\lambda)$ is not hyperfinite$\}$ is $\mu$-measurable [**8**, Theorem 1], so that $\mathscr{A}$ can be written as a direct sum of the two types considered.

Suppose first that $\mathscr{A}(\lambda)\cong\mathscr{F}$ for every $\lambda$. Replacing $\mathscr{A}$ by $\mathscr{A}\otimes\boldsymbol{C}$ if necessary, we may assume that $\mathscr{A}$ is of type $\mathrm{II}_{1,\infty}$, and, since isomorphisms of type $\mathrm{II}_{1,\infty}$ factors are spatial, we may further assume that $\mathscr{A}=\int_\Lambda \oplus (\mathscr{F}\otimes\boldsymbol{C})\mu(d\lambda)$. Since $\mathscr{F}$ (and hence $\mathscr{F}\otimes\boldsymbol{C}$) is not normal (see [**1**, p. 307, Exercise 12d]), it follows from Corollary 2 that, if $\mathscr{B}\subset\mathscr{F}\otimes\boldsymbol{C}$ is a nonnormal subalgebra, then $\mathscr{B}_1=\int_\Lambda \oplus \mathscr{B}\mu(d\lambda)$ is nonnormal in $\mathscr{A}$.

Suppose next that no $\mathscr{A}(\lambda)$ is hyperfinite. We begin by constructing a maximal full hyperfinite $W$-* subalgebra $\mathscr{B}$ of $\mathscr{A}$ such that $\mathscr{B}\cap\mathscr{B}'=\mathscr{Z}$. Recall that, if $\mathscr{C}$ is any finite $W$-* algebra with direct integral decomposition $\mathscr{C}=\int\oplus\mathscr{C}(\lambda)\mu(d\lambda)$, we can define, for $T\in\mathscr{C}, T^\natural=\int_\Lambda\oplus \mathrm{tr}(T(\lambda))\mu(d\lambda)$ the canonical central trace and $|T|_2$ the trace norm, so that if $\{T_n\}\subset\mathscr{C}$ is a bounded sequence, $T_n\to T$ strongly if and only if $|T_n-T|_2\to 0$ [**6**, Definition 2.6, Corollary 2.15, and Lemma 2.16]. The following result, whose proof applies in our case, is [**1**, p. 289, Lemma 2].

LEMMA 4. *If $\mathscr{C}$ is a finite $W$-* algebra and $T^\natural$ denotes the central trace of $T\in\mathscr{C}$, then, for $T\in\mathscr{C}$ and $\varepsilon>0$ such that $|[T, S]|_2\leqq\varepsilon|S|$ for all $S\in\mathscr{C}$ ($[T, S]=TS-ST$), we have $|T-T^\natural|_2\leqq\varepsilon$.*

We apply this lemma as follows, where $\mathscr{A}$ is again a $\mathrm{II}_1$ algebra with no $\mathscr{A}(\lambda)$ isomorphic to $\mathscr{F}$

LEMMA 5. *The set $\mathscr{H}=\{\mathscr{B}|\mathscr{B}$ is a hyperfinite $W$-* subalgebra of $\mathscr{A}$ such that $\mathscr{B}\cap\mathscr{B}'=\mathscr{Z}\}$, ordered by set inclusion, contains a maximal element.*

PROOF. Let $\mathscr{B}_\alpha$ by any increasing chain in $\mathscr{H}$, and let $\mathscr{B}$ be the $W$-* algebra generated by the union of the $\mathscr{B}_\alpha$. We show, using Lemma 4, that the center of $\mathscr{B}$ is $\mathscr{Z}$, and (using [**8**, Theorem 2]) we prove in Lemma 6 that $\mathscr{B}$ is hyperfinite. Since $\mathscr{B}$ is an upper bound for the chain $\mathscr{B}_\alpha$ and $\mathscr{B} \in \mathscr{H}$, the result then follows from Zorn's lemma.

Since $H$ is separable, it is easy to see that, given $S \in \mathscr{B} \cap \mathscr{B}'$, there is a sequence $S_n \in \mathscr{B}_{\alpha_n}$ such that $S_n \to S$ strongly and (by the Kaplansky density theorem) such that $|S_n| \leqq |S|$. Hence $|S_n - S|_2 \to 0$. Given any $\varepsilon > 0$, there is $S_n$ such that $|S_n - S|_2 < \varepsilon/2$. Thus $|[S_n, T]|_2 = |[S_n - S, T]|_2 \leqq 2\,|S_n - S|_2\,|T| \leqq \varepsilon\,|T|$ for all $T \in \mathscr{B}$ and a fortiori for all $T \in \mathscr{B}_{\alpha_n}$. By Lemma 4 we have $|S_n - S_n^\natural|_2 \leqq \varepsilon$. Thus $|S - S_n^\natural|_2 \leqq |S - S_n|_2 + |S_n - S_n^\natural|_2 \leqq 2\varepsilon$, whence $S \in \mathscr{Z}$. Thus $\mathscr{B} \cap \mathscr{B}' = \mathscr{Z}$. Q.E.D.

Before proving that $\mathscr{B}$ is hyperfinite, we establish the following lemma, in which $[\mathscr{C}_n]$ denotes the $W$-* algebra generated by a sequence of $W$-* algebras $\mathscr{C}_n$.

LEMMA 6. *If $\mathscr{C}$ is generated by the union of an increasing sequence of $W$-* subalgebras $\mathscr{C}_n$ such that $\mathscr{C}_n \cap \mathscr{C}_n' = \mathscr{Z} = \mathscr{C} \cap \mathscr{C}'$ for each $n$, then*

$$\mathscr{C} = \int_\Lambda \oplus\, [\mathscr{C}_n(\lambda)]\mu(d\lambda), \quad \textit{where } \mathscr{C}_n = \int_\Lambda \oplus\, \mathscr{C}_n(\lambda)\mu(d\lambda)$$

*for each $n$.*

PROOF. For each $n$ let $T_{n,m} = \int_\Lambda \oplus\, T_{n,m}(\lambda)\mu(d\lambda)$ generate $\mathscr{C}_n(\lambda)$ $\mu$-a.e. Then for $\mu$-a.e. $\lambda$, $[\mathscr{C}_n(\lambda)]$ is the $W$-* algebra generated by all the operators $T_{n,m}(\lambda)$, and $\int_\Lambda \oplus\, [\mathscr{C}_n(\lambda)]\mu(d\lambda)$ is the $W$-* algebra generated by the operators $T_{n,m}$ together with $\mathscr{Z}$. On the other hand, each $\mathscr{C}_n$ is generated by $T_{n,m}$ and by $\mathscr{Z}$, whence $\mathscr{C}$ is generated by all the $T_{n,m}$ and $\mathscr{Z}$. Q.E.D.

Now since $\mathscr{B} \cap \mathscr{B}' = \mathscr{Z}$, $\mathscr{B}$ is decomposable. Moreover, $\mathscr{B}$ is generated by a sequence $T_n$ in the unit sphere of $\mathscr{B}$. By the Kaplansky density theorem, for each $n$ there are a sequence $\alpha_{n,m}$ and a sequence $T_{n,m}$ in $\mathscr{B}_{\alpha_{n,m}}$ such that $|T_{n,m} - T_n|_2 \to 0$. By the total ordering of the $\mathscr{B}_\alpha$, $\mathscr{B}$ is generated by an increasing sequence of hyperfinite algebras $\mathscr{B}_n = \mathscr{B}_{\alpha_n}$. By Lemma 6, $\mathscr{B} = \int_\Lambda \oplus\, [\mathscr{B}_n(\lambda)]\mu(d\lambda)$. Since each $\mathscr{B}_n$ is hyperfinite, $\mathscr{B}_n(\lambda)$ is hyperfinite for all $n$ and for $\mu$-a.e. $\lambda$ by [**8**, Theorem 2]. Hence $[\mathscr{B}_n(\lambda)]$ is hyperfinite for $\mu$-a.e. $\lambda$, and therefore $\mathscr{B}$ is hyperfinite [**8**, Theorem 2]. This proves Lemma 5. Q.E.D.

Let $\mathscr{B} = \int_\Lambda \oplus\, \mathscr{B}(\lambda)\mu(d\lambda)$ be a maximal element of $\mathscr{H}$. We assert that for $\mu$-a.e. $\lambda$, $\mathscr{B}(\lambda)$ is a maximal hyperfinite subfactor of $\mathscr{A}(\lambda)$. Assume for the moment that this is proved. By [**2**, Theorem 3], $\mathscr{B}(\lambda)^{cc} \neq \mathscr{B}(\lambda)$ for $\mu$-a.e. $\lambda$. Hence by Corollary 2, $\mathscr{B}$ is not normal in $\mathscr{A}$.

To establish our assertion, we shall show that the set $\mathscr{M}' = \{\lambda \mid \mathscr{B}(\lambda)$ is not maximal$\}$ is $\mu$-measurable, and that if $\mu(\mathscr{M}') \neq 0$ then $\mathscr{B}$ is not maximal in $\mathscr{A}$.

Observe first that $\mathscr{B}(\lambda)$ fails to be maximal if and only if for every $n$ there is a set of matrix units $E_{i,j}^n$ in $\mathscr{A}(\lambda)$ such that the type I factors they generate form an increasing sequence whose union generates a hyperfinite type $\mathrm{II}_1$ subfactor of $\mathscr{A}(\lambda)$ [**2**, Theorem 2] containing both $\mathscr{B}(\lambda)$ and some $T \notin \mathscr{B}(\lambda)$.

To apply this discussion, let $\mathscr{S}_n$ denote the Cartesian product of $n^2$ copies of $\mathscr{S}$, with a typical element of $\mathscr{S}_n$ denoted by $T_{i,j}^n$. Let $C_0$ denote the complex numbers with rational real and imaginary parts. Let $B_n \in \mathscr{B}$ be a sequence of operators such that $\{B_n(\lambda)\}$ are strong-* dense in the unit ball of $\mathscr{B}(\lambda)$ $\mu$-a.e., and such that the $B_n(\lambda)$ are strong-* continuous in $\lambda$. Consider the subset $\mathscr{N}'$ of $\Lambda \times \prod_{n=1}^{\infty} \mathscr{S}_n \times \mathscr{S}$ whose elements $[\lambda, T_{i,j}^n, R]$ satisfy the following conditions:

(a) $T_{i,j}^n \in \mathscr{A}(\lambda)$ for every $n$, $i$, and $j$, and $T_{i,j}^n$ is a set of matrix units for every $n$.

(b) For every $k$ there are an $n$ and coefficients $b_{i,j}^n \in C_0$ such that for $T = \sum_{i,j=1}^n b_{i,j}^n T_{i,j}^n$ we have $T \in \mathscr{S}$ and $|R - T|_2 < 1/k$.

(c) $R \notin \mathscr{B}(\lambda)$.

(d) For every $k$ and every $T_{r,s}^n$ there are coefficients $C_{i,j}^n \in C_0$ such that for $T = \sum_{i,j=1}^{n+1} C_{i,j}^{n+1} T_{i,j}^{n+1}$ we have $T \in \mathscr{S}$ and $|T_{r,s}^n - T|_2 < 1/k$.

(e) For every $k$ and for every $m$ there are an $n$ and coefficients $d_{i,j}^n \in C_0$ such that for $T = \sum_{i,j=1}^n d_{i,j}^n T_{i,j}^n$ we have $T \in \mathscr{S}$ and $|B_m(\lambda) - T|_2 < 1/k$.

It is easy to see that these countably many conditions define $\mathscr{N}'$ as a Borel subset of $\Lambda \times \prod_{n=1}^{\infty} \mathscr{S}_n \times \mathscr{S}$ whose projection onto $\Lambda$ differs by a $\mu$-null set from $\mathscr{M}'$, as follows from our discussion above. Hence $\mathscr{M}'$ is $\mu$-measurable [**4**, Lemma I.4.6]. If moreover $\mu(\mathscr{M}') \neq 0$, we use [**4**, Lemma I.4.7] to construct functions $T_{i,j}^n(\lambda)$ and $R(\lambda)$ such that

$$[\lambda, T_{i,j}^n(\lambda), R(\lambda)] \in \mathscr{N}'$$

for $\mu$-a.e. $\lambda$ in $\mathscr{M}'$. Letting $\mathscr{C}(\lambda)$ be the hyperfinite type $\mathrm{II}_1$ factor generated by the $T_{i,j}^n(\lambda)$, it is clear that the hyperfinite subalgebra $\mathscr{C}$ of $\mathscr{A}$ defined by

$$\mathscr{C} = \int_{\Lambda - \mathscr{M}'} \oplus\, \mathscr{B}(\lambda)\mu(d\lambda) + \int_{\mathscr{M}'} \oplus\, \mathscr{C}(\lambda)\mu(d\lambda)$$

contains $\mathscr{B}$ properly, since $0 \oplus R \in \mathscr{C}$ but $0 \oplus R \notin \mathscr{B}$, where

$$R = \int_{\mathscr{M}'} \oplus R(\lambda)\mu(d\lambda).$$

Since the $\mathscr{C}(\lambda)$ are factors, $\mathscr{C} \cap \mathscr{C}' = \mathscr{Z}$. Thus we have a contradiction to the maximality of $\mathscr{B}$, and thus we have established the fact that type $\mathrm{II}_1$ $W$-* algebras are not normal.

We summarize and conclude with the following theorem.

THEOREM 7. *No type* II *$W$-* algebra $\mathscr{A}$ on separable Hilbert space is normal.*

PROOF. We have already established this result for $\mathscr{A}$ of type $\mathrm{II}_1$. In fact, we have shown that a nonnormal subalgebra $\mathscr{B}$ of $\mathscr{A}$ can be chosen so that $\mathscr{B}\cap\mathscr{B}'=\mathscr{Z}$. If $\mathscr{A}$ is of type $\mathrm{II}_\infty$, by [7, Theorem 9] $\mathscr{A}$ is isomorphic to $\mathscr{A}_1\otimes\mathscr{B}(J)$ where $\mathscr{A}_1$ is of type $\mathrm{II}_1$ and $J$ is a separable Hilbert space. If $\mathscr{B}$ is a nonnormal subalgebra of $\mathscr{A}_1$ as constructed above, a direct calculation shows that $\mathscr{B}\otimes\mathscr{B}(J)$ is not normal in $\mathscr{A}$. Since any type II algebra is a direct sum of a type $\mathrm{II}_1$ and a type $\mathrm{II}_\infty$ algebra, the proof is complete. Q.E.D.

## BIBLIOGRAPHY


1. J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien*, Gauthier-Villars, Paris, 1957. MR **20** #1234.
2. B. Fuglede and R. Kadison, *On a conjecture of Murray and von Neumann*, Proc. Nat. Acad. Sci. U.S.A. **37** (1951), 420–425. MR **13,** 255.
3. F. Murray and J. von Neumann, *On rings of operators*, Ann. of Math. **37** (1936), 116–229.
4. J. Schwartz, *W-* algebras*, Gordon and Breach, New York, 1967. MR **38** #547.
5. J. von Neumann, *On rings of operators*. III, Ann. of Math. **41** (1940), 94–161. MR **1,** 146.
6. P. Willig, *Trace norms, global properties, and direct integral decompositions of W-* algebras*, Comm. Pure Appl. Math. **22** (1969), 839–862. MR **42** #5063.
7. ———, *Property* L *and direct integral decompositions of W-* algebras*, Proc. Amer. Math. Soc. **30** (1971), 87–91. MR **44** #3137.
8. ———, *On hyperfinite W-* algebras*, Notices Amer. Math. Soc. **18** (1971), 954.


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