

A NOTE ON TOPOLOGICAL DYNAMICS AND LIMITING EQUATIONS

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ABSTRACT. A theorem on uniform asymptotic stability of the null solution of a system of differential equations is proved while assuming that the null solution of a limiting equation is uniformly asymptotically stable. This generalizes some of the results of L. Markus.

1. The concept of the *set of limiting equations* of a given differential equation has been introduced by G. R. Sell [4]. The notion of *asymptotically autonomous differential equations* introduced by L. Markus [2] can be described as those differential equations for which the set of limiting equations consists of a single point. In [4], G. R. Sell has proved a theorem on asymptotic stability of the null solution of a given differential equation while assuming that the null solution of the given differential equation is uniformly stable and the null solution of every limiting equation is asymptotically stable (in a uniform sense). However, as pointed out in a remark ([4, p. 273] and [5, p. 536]), his theorem does not generalize a result of L. Markus [2, Theorem 2]. The aim of this paper is to prove a theorem on uniform asymptotic stability which generalizes the result of L. Markus [2, Theorem 2].

2. Throughout this paper, we follow the same notation as in [3] and [4]. Let W be an open set in R^n , Euclidean n -space. The Euclidean norm on R^n will be denoted by $|x|$. Let $C = C(W \times R, R^n)$ denote the set of all continuous functions f defined on $W \times R$ with values in R^n . We shall say that a function f is *admissible* [3] if (i) $f \in C$, and (ii) the solutions of the differential equation $x' = f(x, t)$ are unique. By the second condition we mean that given any point (x_0, t_0) in $W \times R$, there is precisely one solution ϕ of $x' = f(x, t)$ that satisfies $\phi(t_0) = x_0$. It is evident that if f is an admissible function, then every translate f_τ of f (where $f_\tau(x, t) = f(x, t + \tau)$) is an admissible function. Also, if f satisfies the global existence property, then so does each f_τ . Let $F = \{f_\tau : \tau \in R\}$ be the space of translates of f ; then F is a

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subset of C . Now let f be an admissible function and consider the space of translates F in the *compact open topology* [1] on C . Let $F_{co}^* = \text{Cl } F$ (that is, the closure in the compact open topology) be the hull of f . We shall say that f is *regular* [3] if every function f^* in the hull F_{co}^* is admissible. It is shown in [3] that the mapping $\pi^*: F_{co}^* \times \mathbb{R} \rightarrow F_{co}^*$, defined by $\pi^*(f^*, t) = f_t^*$, is a dynamical system on F_{co}^* , when F_{co}^* has the compact open topology. We shall say that the motion f_t is *positively compact* [3] if the closure of $\{\pi^*(f, t) : t \geq 0\}$ lies in a compact subset of F_{co}^* .

DEFINITION OF LIMITING EQUATIONS [4]. Let $f \in C$ and let F_{co}^* be the hull of f (neither regularity nor admissibility of f will be important here). Let $\pi^*(f, t) = f_t$ be the flow on F_{co}^* and let Ω_f^* denote the ω -limit set of f in this flow. If the ω -limit set Ω_f^* of f in F_{co}^* is nonempty, then we say that the set of *limiting equations* for

$$(2.1) \quad x' = f(x, t)$$

is the set of all differential equations of the form

$$(2.2) \quad x' = f^*(x, t),$$

where $f^* \in \Omega_f^*$.

We need the following result in our subsequent discussion.

LEMMA 2.1 (KAMKE). *Let $\{g_n\}$ be a sequence of continuous functions in C , and let $g = \lim g_n$, where the convergence is in the compact open topology on C . Let ϕ_n be a solution of $x' = g_n(x, t)$ with $\phi_n(0) \rightarrow x_0 \in W$. Then there is a subsequence of $\{\phi_n\}$ that converges to a solution ϕ of $x' = g(x, t)$ that satisfies $\phi(0) = x_0$, and the convergence is uniform on compact sets in the interval of definition of ϕ . If, in addition, the solutions of $x' = g(x, t)$ are unique, then $\phi = \lim \phi_n$, where the convergence is uniform on compact sets in the interval of definition of ϕ .*

In Theorems 4 and 5 of [4], G. R. Sell has proved that if the equation (2.1), with the assumption that f is a regular function and $f(0, t) = 0$ for all $t \geq 0$, has a "stable" solution, then the limiting equations (2.2) have the same property. The problem of reversing these roles is a bit delicate. That is, if we assume some stability properties of the solutions of (2.2), then it is generally harder to derive results about the given equation (2.1). However, he proved the following result.

THEOREM 2.1 [4, THEOREM 6]. *Let $f \in C$ be a regular function with $f(0, t) = 0$ ($t \geq 0$) and assume that the motion f_t is positively compact, in the compact open topology. If (i) the null solution of (2.1) is uniformly stable, and (ii) the null solution of every limiting equation (2.2) is asymptotically stable (in a uniform sense), that is $|\phi(x_0, f^*, t)| \rightarrow 0$ as $t \rightarrow \infty$ whenever*

$|x_0| \leq a$ and $f^* \in \Omega_f^*$, then the null solution of the given equation (2.1) is asymptotically stable.

3. Now, we shall prove the following main result.

THEOREM 3.1. *Let f be a regular function with $f(0, t) = 0$ ($t \geq 0$). If there exists a function f^* in Ω_f^* such that the null solution of (2.2) is uniformly asymptotically stable, then the null solution of the given equation (2.1) is uniformly asymptotically stable.*

PROOF. Let $\phi(x_0, f^*, t)$ be any solution of (2.2) with $\phi(x_0, f^*, 0) = x_0$. Let $\varepsilon > 0$ be given. Since the null solution of (2.2) is uniformly asymptotically stable, given $\varepsilon/2$ there exists a $\delta_1 = \delta_1(\varepsilon/2) > 0$ (without any loss of generality we can suppose that $\delta_1 < \varepsilon/2$) such that the inequality $|x_0| < \delta_1$ implies

$$(3.1) \quad |\phi(x_0, f^*, t)| < \varepsilon/2 \quad \text{for } t \geq 0,$$

and there exists a $\delta_0 > 0$, such that for every $\eta > 0$ there exists a $T = T(\eta) > 0$ such that the inequality $|x_0| < \delta_0$ implies that

$$(3.2) \quad |\phi(x_0, f^*, t)| < \eta \quad \text{for } t \geq T.$$

Choose $\delta = \min(\delta_1, \delta_0)$. From (3.1) and (3.2), we have

$$(3.3) \quad |\phi(x_0, f^*, t)| < \varepsilon/2 \quad \text{for } t \geq 0,$$

whenever $|x_0| < \delta$, and for $\delta/2$ there exists a $T = T(\delta/2)$ such that the inequality $|x_0| < \delta$ implies

$$(3.4) \quad |\phi(x_0, f^*, t)| < \delta/2 \quad \text{for } t \geq T.$$

Now, by Lemma 2.1, given $f^* \in \Omega_f^*$ and $\delta/2$ there is a $d = d(\delta/2) > 0$ such that

$$|\phi(x_0, f^*, t) - \phi(x_0, g, t)| < \delta/2, \quad t \in [0, T],$$

$|x_0| < \delta$, whenever $\rho(f^*, g) < d$, where ρ is any metric which generates the compact open topology on F_{co}^* . By the definition of Ω_f^* we can find a translate f_r , $\tau \geq 0$, such that $\rho(f^*, f_r) < d$. Therefore

$$(3.5) \quad |\phi(x_0, f^*, t) - \phi(x_0, f_r, t)| < \delta/2 \quad \text{for } t \in [0, T], |x_0| < \delta.$$

Combining (3.3) and (3.5), we get

$$(3.6) \quad |\phi(x_0, f_r, t)| \leq |\phi(x_0, f^*, t)| + |\phi(x_0, f_r, t) - \phi(x_0, f^*, t)| < \varepsilon/2 + \delta/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

That is, $|\phi(x_0, f_r, t)| < \varepsilon$ for $t \in [0, T]$, $|x_0| < \delta$. Now, let $x_1 = \phi(x_0, f_r, T)$; then from (3.4) and (3.5) we have $|x_1| \leq |\phi(x_0, f^*, T)| + |x_1 - \phi(x_0, f^*, T)| < \delta/2 + \delta/2 = \delta$. Therefore, from (3.6) it follows that $|\phi(x_1, f_r, t)| < \varepsilon$ for

$t \in [0, T]$, which in turn implies that $|\phi(x_0, f_r, t)| < \varepsilon$ for $t \in [0, 2T]$, $|x_0| < \delta$. Similarly, if we suppose $x_2 = \phi(x_1, f_r, T)$, then $|x_2| \leq |\phi(x_1, f^*, T)| + |x_2 - \phi(x_1, f^*, T)| < \delta/2 + \delta/2 = \delta$, and again, from (3.6), it follows that $|\phi(x_0, f_r, t)| < \varepsilon$ for $t \in [0, 3T]$, $|x_0| < \delta$. Now, let m be a positive integer and assume that $|\phi(x_0, f_r, t)| < \varepsilon$ for $t \in [0, mT]$, $|x_0| < \delta$, and assume that $|\phi(x_0, f_r, mT)| < \delta$. Let $x_m = \phi(x_0, f_r, mT)$; then from (3.3) and (3.5), we have

$$\begin{aligned} |\phi(x_m, f_r, t)| &\leq |\phi(x_m, f^*, t)| + |\phi(x_m, f_r, t) - \phi(x_m, f^*, t)| \\ &< \varepsilon/2 + \delta/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon, \end{aligned}$$

for $t \in [0, T]$. Therefore, $\phi(x_0, f_r, t)$ can be continued to the interval $[mT, (m+1)T]$ on which $|\phi(x_0, f_r, t)| < \varepsilon$. Let $x_{m+1} = \phi(x_0, f_r, (m+1)T)$; then from (3.4) and (3.5) and the fact that $|x_m| < \delta$, we have $|x_{m+1}| \leq |\phi(x_m, f^*, T)| + |x_{m+1} - \phi(x_m, f^*, T)| < \delta/2 + \delta/2 = \delta$. Thus, by induction, $|\phi(x_0, f_r, t)| < \varepsilon$, whenever $|x_0| < \delta$, on every interval $[mT, (m+1)T]$ and hence on $[0, \infty)$. Since δ is independent of τ , we have, for a given $\varepsilon > 0$ there exist a $\delta = \delta(\varepsilon) > 0$ and a $\tau = \tau(\varepsilon) \geq 0$ such that $|\phi(x_0, f, t)| < \varepsilon$, $t \geq \tau$, provided $|x_0| < \delta$. Now from the continuity of solutions with respect to the initial values and the uniqueness of solutions, it follows that the null solution of (2.1) is uniformly stable. For the rest of the proof, choose $\delta_0 = \delta_0$ and fix $|x_0| < \delta_0$. Let $\eta > 0$ be given. Choose $\bar{\delta}(\eta) = \min(\delta_1(\eta/2), \delta_0)$, $0 < \bar{\delta} < \eta$, and $T_1(\eta) = T(\bar{\delta}/2)$. Let $y_0 = \phi(x_0, f_r, T_1)$, $\tau = \tau(\eta) \geq 0$. Then

$$|y_0| \leq |\phi(x_0, f^*, T_1)| + |\phi(x_0, f_r, T_1) - \phi(x_0, f^*, T_1)| < \bar{\delta}/2 + \bar{\delta}/2 = \bar{\delta}.$$

Thus by the first part of the proof, $|\phi(y_0, f, t)| < \eta$ for $t \geq \tau$, whenever $|y_0| < \bar{\delta}(\eta)$. Now by the uniqueness of solutions, it follows that $|\phi(x_0, f, t)| < \eta$ for $t \geq T^*$, whenever $|x_0| < \delta_0$, where $T^* = T^*(\eta) = \tau + T_1$. This completes the proof of the theorem.

REMARK 3.1. Observe that in Theorem 2.1, G. R. Sell assumed that the motion f_t is positively compact and the null solution of every limiting equation (2.2) is asymptotically stable (in a uniform sense) to prove that the null solution of (2.1) is asymptotically stable. Clearly, Theorem 2.1 does not generalize a result of L. Markus [2, Theorem 2] as remarked in [4, p. 273] and [5, p. 536]. It is appropriate to remark here that, in Theorem 3.1, we merely assume that the null solution of (2.2) for some f^* in Ω_f^* is uniformly asymptotically stable to prove that the null solution of (2.1) is uniformly asymptotically stable. Obviously, this would generalize the result of L. Markus [2, Theorem 2] for asymptotically autonomous equations.

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