PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 40, Number 1, September 1973

SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

A CHARACTERIZATION OF REALCOMPACT EXTENSIONS

MARLON RAYBURN

Let X be a Tychonov space, R be the real numbers and $R^*=R\cup\{\infty\}$ be its one point compactification. As in [1, 8B], for each $g\in C(X)$, let $g_*:\beta X\to R^*$ be its Stone extension and $v_gX=\{p\in\beta X:g_*(p)\neq\infty\}=g_*^{\leftarrow}[R]$.

Let T be any Tychonov space containing X densely and let $f:\beta X\to\beta T$ be the continuous map fixing X pointwise. Let $C_T\subseteq C(X)$ be that family of functions which can be continuously extended to T. In [2, Theorem 2], it was shown that $g\in C_T$ if and only if for every sequence $(F_n)_{n=1}^{\infty}$ of closed sets in R such that $\bigcap_{n=1}^{\infty} F_n=\varnothing$, it follows that $\bigcap_{n=1}^{\infty}\operatorname{cl}_T g^{\leftarrow}[F_n]=\varnothing$.

LEMMA. *For any such* $T, f^{\leftarrow}[T]\subseteq\bigcap_{g\in C_T} v_gX$.

PROOF. Choose any $g\in C_T$, extend it to $G\in C(T)$ and let $G_*:\beta T\to R^*$ be its Stone extension. Set $h:\beta X\to R^*$ by $h=G\circ f$. But $h|X=g$, so h and g_* agree on a dense subset of βX, whence $h=g_*$. Clearly $h|f^{\leftarrow}[T]$ takes $f^{\leftarrow}[T]$ into R, so $g_*^{\leftarrow}[R]=v_gX\supseteq f^{\leftarrow}[T]$.

THEOREM. *T is realcompact if and only if* $f^{\leftarrow}[T]=\bigcap_{g\in C_T} v_gX$.

PROOF. (*Only if*) Let $p\in\beta X-f^{\leftarrow}[T]$. Then $f(p)\in\beta T-T$. Now T is realcompact, so there is a $G\in C(T)$ such that $G_*:\beta T\to R^*$ and $G_*(f(p))=\infty$. Let $G|X=g\in C(X)$ have Stone extension $g_*:\beta X\to R^*$. Define h: $\beta X\to R^*$ by $h=G_*\circ f$. Then h is continuous and $h|X=g=g_*|X$, so $g_*=G_*\circ f$. Thus $g_*(p)=G_*(f(p))=\infty$. Since $G\in C(T)$, $g\in C_T$ and $p\notin v_gX$.

(*If*) Let $p\in\beta X-f^{\leftarrow}[T]$. Then there is a $g\in C_T$ such that $p\notin v_gX$, so $g_*(p)=\infty$. Now let $G\in C(T)$ be such that $G|X=g$. Extend g to G_*: $\beta T\to R^*$ and compare $g_*:\beta X\to R^*$ with $G_*\circ f:\beta X\to R^*$. These agree on

Presented to the Society, February 2, 1972; received by the editors May 30, 1972.
AMS (MOS) subject classifications (1970). Primary 54D60.

X, so $g_*=G_* \circ f$. Thus $G_*(f(p))=\infty$. So given any $t \in \beta T-T$, there is a $G \in C(T)$ such that $G_*(t)=\infty$, whence T is realcompact.

COROLLARY. *For any extension* T, $\bigcap_{g\in C_T} v_g X=f^{\leftarrow}[vT]$.

PROOF. $C_T=C_{vT}$.

REFERENCES

1. L. Gillman and M. Jerison, *Rings of continuous functions*, University Series in Higher Math., Van Nostrand, Princeton, N.J., 1960. MR **22** #6994.

2. R. Engelking, *Remarks on real-compact spaces*, Fund. Math. **55** (1964), 303–308. MR **31** #4000.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MANITOBA, WINNIPEG, MANITOBA, CANADA