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ON A PROBLEM OF F. RIESZ CONCERNING PROXIMITY STRUCTURES

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ABSTRACT. It is shown that every separated Lodato proximity is induced by the elementary proximity on a T_1 bicompactification of the original space.

A *basic proximity structure* Π on a set X is a relation on $\mathfrak{P}(X)$ satisfying the following requirements

$P_1: \Pi = \Pi^{-1}$,

$P_2: A \cup B \in \Pi(C) \Leftrightarrow A \in \Pi(C)$ or $B \in \Pi(C)$,

$P_3: A \cap B \neq \varnothing \Rightarrow A \in \Pi(B)$,

$P_4: \varnothing \notin \Pi(A)\ \forall A \in \mathfrak{P}(X)$.

Here $\Pi(A) = [B: \langle A, B\rangle \in \Pi]$. Each proximity structure induces a closure operator on X as follows: $c_\pi(A) = [x: [x] \in \Pi(A)]$. If for a proximity relation the additional condition

$P_5: c_\pi(A) \in \Pi(B) \Rightarrow A \in \Pi(B)$

also holds then Π is called a *LO-proximity*. A relation Π is said to be *separated* if in it

$P_6: [x] \in \Pi([y]) \Leftrightarrow x = y$,

is valid.

A *grill* on X is a family $\mathfrak{G} \subset \mathfrak{P}(X)$ which satisfies:

$G_1: A \supset B \in \mathfrak{G} \Rightarrow A \in \mathfrak{G}$,

$G_2: A \cup B \in \mathfrak{G} \Rightarrow A \in \mathfrak{G}$ or $B \in \mathfrak{G}$,

$G_3: \varnothing \notin \mathfrak{G}$.

Grills were introduced by Choquet [1] in 1947. It is known (see for example [5, Lemma 5.7]) that every grill is the union of ultrafilters. It is easy to verify that the converse also holds. It is an immediate consequence of P_2 and P_4 that $\Pi(A)$ is a grill for all $A \in \mathfrak{P}(X)$.

Define $b(\Pi, \mathfrak{G}) = [B: c_\pi(B) \in \mathfrak{G}]$. One easily verifies that if $\mathfrak{G}$ is a grill then $b(\Pi, \mathfrak{G})$ is a grill, $b(\Pi, \mathfrak{G}) \supset \mathfrak{G}$, and $\mathfrak{G}_1 \supset \mathfrak{G}_2$ implies $b(\Pi, \mathfrak{G}_1) \supset b(\Pi, \mathfrak{G}_2)$.

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A grill $\mathfrak{G}$ for which it is true that $A, B \in \mathfrak{G} \Rightarrow A \in \Pi(B)$ will be called a Π-*clan*. A Π-clan $\mathfrak{G}$ which satisfies the additional condition $b(\Pi, \mathfrak{G}) = \mathfrak{G}$ is called a *bunch*.

Let c be a Čech closure operator on X then the relation Π_0 on X defined by

$$A \in \Pi_0(B) \Leftrightarrow c(A) \cap c(B) \neq \varnothing$$

is a basic proximity. It is called the *elementary proximity* associated with c. It is not in general true that $c = c_{\pi_0}$. However, if Π is a separated LO-proximity then c_π is the closure operator for a T_1-topology and if c is a Kuratowski closure operator which generates a T_1-topology then $c = c_{\pi_0}$.

Let two proximity spaces (X, Π) and (Y, Π^*) and an injection $\varphi : X \to Y$ be given. Then Π is said to be induced by Π^* if

$$A \in \Pi(B) \Leftrightarrow \varphi(A) \in \Pi^*(\varphi(B)).$$

The problem of Riesz [**6**], referred to in the title, is the following: what types of proximity can be induced by elementary proximities on suitably constructed extension spaces of the original space? Riesz posed the problem in 1908, suggested a possible approach (using maximal Π-clans) but gave no answer. Clearly, the problem suggests that there may be a close relation between the proximities compatible with a given topological space and a certain class of topological extensions of the space. For EF-proximities Smirnov [**7**] in 1952 showed that they are induced by elementary proximities on T_2-bicompactifications of the underlying space. Improving on earlier work of Leader [**3**] and Lodato [**4**] Gagrat and Naimpally [**2**] recently showed that every separated LO-proximity which satisfies the additional condition:

GN: Given $A \in \Pi(B)$ there exists a bunch $\mathfrak{B}$ such that $A, B \in \mathfrak{B}$, is induced by the elementary proximity on a T_1-bicompactification of the original space.

We shall show that every LO-proximity satisfies GN (Theorem 4) and hence every separated LO-proximity can be induced by an elementary proximity. Harris has coined the name WI-proximity for those proximities which can be induced by an elementary proximity. He has shown that every separated WI-proximity is a LO-proximity. It now follows that the separated WI-proximities are exactly the separated LO-proximities.

The result stated above is the final link in a chain whose other members are also of interest.

THEOREM 1. *Let* $\mathfrak{F}$ *be a filter and* Π *a basic proximity on* X; *then* $\Pi(\mathfrak{F}) = \bigcap [\Pi(A) : A \in \mathfrak{F}]$ *is a grill.*

PROOF. Clearly $\Pi(\mathfrak{F})$ satisfies conditions G_1 and G_3. Now assume $A \cup B \in \Pi(\mathfrak{F})$ and $A \notin \Pi(\mathfrak{F})$, $B \notin \Pi(\mathfrak{F})$. Then there exist sets C and D in $\mathfrak{F}$ such that $A \notin \Pi(C)$, $B \notin \Pi(D)$. From this $A \notin \Pi(C \cap D)$, $B \notin \Pi(C \cap D)$ follows. Since $C \cap D \in \mathfrak{F}$ we have $A \cup B \in \Pi(C \cap D)$ and thus a contradiction to the fact that $\Pi(C \cap D)$ is a grill.

THEOREM 2. *Let* Π *be a basic proximity on* X *then* $A \in \Pi(B)$ *implies the existence of a* Π*-clan* $\mathfrak{G}$ *on* X *such that* $A, B \in \mathfrak{G}$.

PROOF. Since $\Pi(B)$ is a grill it is a union of ultrafilters. Hence there exists an ultrafilter $\mathfrak{U}_A$ such that $A \in \mathfrak{U}_A \subset \Pi(B)$. It follows from the symmetry of Π that $B \in \Pi(\mathfrak{U}_A)$. Since $\Pi(\mathfrak{U}_A)$ is a grill it follows that there exists an ultrafilter $\mathfrak{U}_B$ such that $B \in \mathfrak{U}_B \subset \Pi(\mathfrak{U}_A)$. Since $\mathfrak{U}_B \subset \Pi(\mathfrak{U}_A)$ implies $\mathfrak{U}_A \subset \Pi(\mathfrak{U}_B)$ a desired Π-clan is $\mathfrak{G} = \mathfrak{U}_A \cup \mathfrak{U}_B$.

THEOREM 3. *Let* Π *be a LO-proximity on* X *then every maximal* Π*-clan is a bunch with respect to* Π.

PROOF. If $\mathfrak{G}$ is a Π-clan than $b(\Pi, \mathfrak{G})$ is a Π-clan. To see this note that since Π is a *LO*-proximity it satisfies P_5 and hence $b(\Pi, \Pi(A)) = \Pi(A)$. Since $\mathfrak{G}$ is a Π-clan we have $\mathfrak{G} \subset \Pi(A)$ for all $A \in \mathfrak{G}$. Hence $b(\Pi, \mathfrak{G}) \subset b(\Pi, \Pi(A)) = \Pi(A)$. By symmetry of Π, $\mathfrak{G} \subset \Pi(B)$ for all $B \in b(\Pi, \mathfrak{G})$ and hence $b(\Pi, \mathfrak{G}) \subset \Pi(B)$ for all $B \in b(\Pi, \mathfrak{G})$. It follows that $b(\Pi, \mathfrak{G})$ is a Π-clan. For every maximal Π-clan $\mathfrak{G}^*$ we then have $\mathfrak{G}^* = b(\Pi, \mathfrak{G}^*)$ (since $b(\Pi, \mathfrak{G}) \supset \mathfrak{G}$ for all grills $\mathfrak{G}$). That is $\mathfrak{G}^*$ is a bunch.

THEOREM 4. *Let* Π *be a LO-proximity on* X *and let* $A \in \Pi(B)$. *Then there exists a bunch* $\mathfrak{B}$ *containing* A *and* B.

PROOF. Let $\mathfrak{G}$ be a Π-clan. There exists a maximal Π-clan $\mathfrak{G}^*$ containing $\mathfrak{G}$. This is proved using Zorn's lemma. By Theorem 3 $\mathfrak{G}^*$ is a bunch. By Theorem 2 a $\mathfrak{G}$ can be found to contain A and B, hence $\mathfrak{G}^*$ contains the two sets.

A more extensive discussion of the ideas employed here is given in a forthcoming article [8] by the author.

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