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NONLINEAR OSCILLATION OF A SUBLINEAR DELAY EQUATION OF ARBITRARY ORDER

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ABSTRACT. The equations considered generalize

$$x^{(n)}(t) + p(t)|x(g(t))|^{\alpha} \operatorname{sgn} x(g(t)) = 0, \qquad 0 < \alpha < 1.$$

A necessary and sufficient condition is established that all solutions are oscillatory when n is even and are either oscillatory or strongly monotone when n is odd. The result makes clear a difference in oscillatory property between sublinear delay equations and the corresponding ordinary differential equations.

We consider the nonlinear delay equation

$$x^{(n)}(t) + p(t)f(t, x(t), x(g(t))) = 0, \tag{1}$$

where the following conditions are always assumed to hold:

(a) $p(t)$ is continuous and nonnegative on $R_+ = [0, \infty)$;

(b) $g(t)$ is continuous on R_+ and such that $g(t) \leqq t$, $\lim_{t\to\infty} g(t) = \infty$;

(c) $f(t, x, y)$ is continuous on $S = R_+ \times R \times R$ and such that $yf(t, x, y) > 0$ for $(t, x, y) \in S$ with $y \neq 0$.

We tacitly assume that under the initial condition

$$x(t) = \phi(t), \quad t \leqq t_0 \quad \text{and} \quad x^{(j)}(t_0) = x_j^0, \qquad j = 1, \cdots, n-1,$$

equation (1) has a solution which can be continued to $[t_0, \infty)$.

A nontrivial solution $x(t)$ of (1) is called oscillatory if there exists a sequence $\{t_k\}_{k=1}^{\infty}$ such that $x(t_k) = 0$ for all k and $\lim_{k\to\infty} t_k = \infty$. Otherwise, a solution is called nonoscillatory. A nonoscillatory solution is said to be strongly monotone if it tends monotonically to zero as $t \to \infty$ together with its first $n-1$ derivatives.

The object of this paper is to establish under appropriate restrictions on f a necessary and sufficient condition that every solution of (1) be oscillatory in the case n is even and be either oscillatory or strongly monotone in the case n is odd. Our theorem generalizes to arbitrary $n \geqq 2$ those of Gollwitzer **[2]** and Ševelo and Odarič **[10]** for the second order delay

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equation $x''(t)+p(t)[x(g(t))]^\alpha=0$, where α is the ratio of odd positive numbers and $\alpha<1$. (We note that Gollwitzer's theorem has been extended to a class of second order functional differential equations by Burkowski [**1**].) Our results show how the rate of growth for large t of the retarded argument $g(t)$ affects the oscillatory property of delay equations in question.

THEOREM 1. *Suppose there exist positive constants K and $\alpha<1$ such that*

$$|f(t,x,y)| \leqq K|y|^\alpha \quad \textit{for } (t,x,y)\in S. \tag{2}$$

Then a necessary condition that every solution of (1) *be oscillatory if n is even and be either oscillatory or strongly monotone if n is odd is that*

$$\int^\infty [g(t)]^{\alpha(n-1)}p(t)\,dt = \infty. \tag{3}$$

PROOF. The proof is based on the arguments developed by Waltman [**13**], Hallam [**3**], Singh [**12**] and Ladas [**7**].

We assume that (3) does not hold and demonstrate that equation (1) has a nonoscillatory solution $x(t)$ such that $\lim_{t\to\infty} x(t)/t^{n-1}=a\neq 0$.

Choose t_0 so large that $g(t)>0$ for $t\geqq t_0\geqq 1$ and integrate (1) n times from t_0 to t. Then we have

$$x(t) = \sum_{j=0}^{n-1}\frac{x^{(j)}(t_0)}{j!}(t-t_0)^j - \frac{1}{(n-1)!}\int_{t_0}^t (t-s)^{n-1}p(s)f(s,x(s),x(g(s)))\,ds,$$

which yields, in view of (2),

$$|x(t)| \leqq Ct^{n-1} + Kt^{n-1}\int_{t_0}^t p(s)\,|x(g(s))|^\alpha\,ds, \qquad t\geqq t_0, \tag{4}$$

where C is a positive constant.

We define the function $F(t)$ by

$$F(t) = C + K\int_{t_0}^t p(s)\,|x(g(s))|^\alpha\,ds. \tag{5}$$

Then

$$|x(t)| \leqq t^{n-1}F(t), \qquad t\geqq t_0. \tag{6}$$

If we choose $t_1\geqq t_0$ so large that $g(t)\geqq t_0$ for $t\geqq t_1$, it follows from (6) and the increasing character of $F(t)$ that

$$|x(g(t))| \leqq [g(t)]^{n-1}F(g(t)) \leqq [g(t)]^{n-1}F(t), \qquad t\geqq t_1. \tag{7}$$

From (5) and (7) we have

$$F'(t) = Kp(t)\,|x(g(t))|^\alpha \leqq Kp(t)[g(t)]^{\alpha(n-1)}F(t)^\alpha, \qquad t\geqq t_1,$$

and consequently,

$$F(t) \leqq \left\{F(t_1)^{1-\alpha} + (1-\alpha)K\int_{t_1}^{t}[g(s)]^{\alpha(n-1)}p(s)\,ds\right\}^{1/(1-\alpha)}$$
$$\leqq \left\{F(t_1)^{1-\alpha} + (1-\alpha)K\int_{t_1}^{\infty}[g(s)]^{\alpha(n-1)}p(s)\,ds\right\}^{1/(1-\alpha)} \equiv C_1,$$

where C_1 is a finite positive constant. The inequalities (6) and (7) then become

$$(6') \qquad |x(t)| \leqq C_1 t^{n-1}, \qquad t \geqq t_1,$$
$$(7') \qquad |x(g(t))| \leqq C_1[g(t)]^{n-1}, \qquad t \geqq t_1.$$

Now we integrate (1) from t_1 to t to obtain

$$x^{(n-1)}(t) = x^{(n-1)}(t_1) - \int_{t_1}^{t} p(s)f(s, x(s), x(g(s)))\,ds,$$

from which and in view of (2), (3), (7′) we conclude that the finite limit $\lim_{t\to\infty} x^{(n-1)}(t)$ exists. If we require that

$$x^{(n-1)}(t_1) > C_1K\int_{t_1}^{\infty}[g(s)]^{\alpha(n-1)}p(s)\,ds,$$

then this limit of $x^{(n-1)}(t)$ is not zero, and the solution $x(t)$ has the desired asymptotic property.

THEOREM 2. *Suppose there exist positive constants k and $\beta<1$ such that*

$$(8) \qquad |f(t, x, y)| \geqq k\,|y|^{\beta} \quad \text{for } (t, x, y) \in S.$$

If

$$(9) \qquad \int^{\infty}[g(t)]^{\beta(n-1)}p(t)\,dt = \infty,$$

then every solution of (1) *is oscillatory in the case n is even and is either oscillatory or strongly monotone in the case n is odd.*

The following lemma of Kiguradze [4] will be needed.

LEMMA. *If $x(t), x'(t), \cdots, x^{(n-1)}(t)$ are absolutely continuous and of constant sign on the interval $[t_0, \infty)$, and $x^{(n)}(t)x(t)\leqq 0$, then there exists an integer l, $0\leqq l\leqq n-1$, which is even if n is odd and odd if n is even, such that*

$$|x(t)| \geqq \frac{(t-t_0)^{n-1}}{(n-1)\cdots(n-l)}\,|x^{(n-1)}(2^{n-l-1}t)|, \qquad t \geqq t_0.$$

PROOF OF THEOREM 2. Our proof is an adaptation of the arguments developed by Ryder and Wend [9] for the case $g(t)\equiv t$ and is similar to that used by Ševelo and Vareh [11] for even order linear delay equations.

Let n be even and let $x(t)$ be a nonoscillatory solution of (1). We may assume that $x(t)>0$ for large t. From the fact that $x^{(n)}(t)<0$ for large t, it follows that $x^{(n-1)}(t)$ is decreasing and that the derivatives of $x(t)$ of orders up to $n-1$ are eventually of constant sign, the odd order derivatives being eventually positive. In particular, $x'(t)>0$, so that $x(t)$ is increasing for large t. According to Kiguradze's lemma we have

$$x(t) \geqq x(2^{l-n+1}t) \geqq \frac{2^{(l-n+1)(n-1)}(t-t_0)^{n-1}}{(n-1)\cdots(n-l)}\, x^{(n-1)}(t)$$

for $t\geqq t_0$, provided t_0 is sufficiently large. Therefore,

$$x(t) \geqq At^{n-1}x^{(n-1)}(t), \qquad t \geqq t_1 = 2t_0, \tag{10}$$

where $A=2^{(l-n+1)(n-1)}/(n-1)\cdots(n-l)$. Since $\lim_{t\to\infty} g(t)=\infty$, there is a $t_2\geqq t_1$ such that $g(t)\geqq t_1$ for $t\geqq t_2$. From (10) and the decreasing character of $x^{(n-1)}(t)$ we then have

$$x(g(t)) \geqq A[g(t)]^{(n-1)}x^{(n-1)}(t), \qquad t \geqq t_2. \tag{11}$$

Combining (1) with (8) and (11) gives

$$x^{(n)}(t) + A^{\beta}p(t)[g(t)]^{\beta(n-1)}[x^{(n-1)}(t)]^{\beta} \leqq 0.$$

Dividing by $[x^{(n-1)}(t)]^{\beta}$ and integrating form t_2 to t we obtain

$$\left.\frac{[x^{(n-1)}(s)]^{1-\beta}}{1-\beta}\right|_{t_2}^{t} + A^{\beta}\int_{t_2}^{t}[g(t)]^{\beta(n-1)}p(s)\,ds \leqq 0,$$

which implies $\int_{t_2}^{\infty}[g(t)]^{\beta(n-1)}p(t)\,dt<\infty$, a contradiction.

The case where $x(t)<0$ for large t can be treated similarly.

Let n be odd and assume the existence of a nonoscillatory solution $x(t)$. If $x(t)$ does not approach zero as $t\to\infty$, then, writing

$$|x(t)| = |x(t)/x(2^{l-n+1}t)| \cdot |x(2^{l-n+1}t)|,$$

applying Kiguradze's formula to $|x(2^{l-n+1}t)|$ and using the decreasing character of $|x^{(n-1)}(t)|$, we have

$$|x(t)| \geqq MAt^{n-1}\,|x^{(n-1)}(t)|, \qquad t \geqq t_1,$$

and

$$|x(g(t))| \geqq MA[g(t)]^{n-1}\,|x^{(n-1)}(t)|, \qquad t \geqq t_2$$

where $M=\inf_{t\geqq t_0}|x(t)/x(2^{l-n+1}t)|$. The proof now proceeds exactly as in the case of even n. Thus it follows that a nonoscillatory solution of (1), if it exists, must approach zero as $t \to \infty$. In this case, not only $x(t)$ but also its first $n-1$ derivatives tend monotonically to zero as $t \to \infty$.

REMARK. Under some additional smoothness assumptions on $g(t)$, oscillation criteria of the form (9) were obtained by Ševelo and Odarič [**10**] for second order equations and by the present authors ([**5**], [**6**]) for higher order equations.

Combining Theorems 1 and 2 we obtain the following

THEOREM 3. *Suppose there exist positive constants k, K, $\alpha < 1$ such that*

$$k\,|y|^\alpha \leqq |f(t, x, y)| \leqq K\,|y|^\alpha \quad \textit{for } (t, x, y) \in S.$$

Then a necessary and sufficient condition that every solution of (1) *be oscillatory when n is even and be either oscillatory or strongly monotone when n is odd is that* (3) *be valid.*

REMARK. If $g(t)$ is of the form $g(t) = t - \tau(t)$ with $0 \leqq \tau(t) \leqq M$, then (3) is equivalent to

$$\int^\infty t^{\alpha(n-1)} p(t)\,dt = \infty. \tag{12}$$

Thus Theorem 3 is an extension of a theorem of Gollwitzer [**2**, Theorem 2] for the second order sublinear delay equation.

REMARK. On the basis of Theorem 3 we can compare the oscillatory property of sublinear delay equations with that of the corresponding differential equations without delay. As an illustration we consider

$$x^{(n)}(t) + p(t)\,|x(t)|^\alpha \operatorname{sgn} x(t) = 0 \tag{13}$$

and

$$x^{(n)}(t) + p(t)\,|x(g(t))|^\alpha \operatorname{sgn} x(g(t)) = 0, \tag{14}$$

where n is even and $0 < \alpha < 1$. It is well known ([**4**], [**8**]) that all solutions of (13) are oscillatory if and only if (12) holds. Therefore, if $g(t)$ is such that the integrals in (3) and (12) converge or diverge simultaneously, e.g., if $\lim_{t \to \infty} g(t)/t = c > 0$, then equations (13) and (14) have the same oscillatory property. It may happen that (12) holds but (3) does not. In this case, all solutions of (13) are oscillatory, while among solutions of (14) there is a nonoscillatory solution.

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