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THE NUMBER OF CONTINUA

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ABSTRACT. It is shown there are precisely 2^n topologically distinct continua of weight n and power m where $p \leqq n \leqq m$ and p is the smallest cardinal for which there is a continuum of power m and weight p. In particular, there are precisely 2^m topologically distinct continua of power m.

All spaces considered are assumed to be Hausdorff and all cardinals infinite. A continuum is a compact connected space. It follows from a classical result of P. Alexandroff and P. Urysohn ([**1**, p. 105]: *The weight of a compact space never exceeds its power*) that there are at most 2^m topologically distinct compact spaces of power m. (There are just 2^m collections, of cardinality at most m, of subsets of a set of cardinality m.) The following sharper result will be useful.

PROPOSITION 1. *There are at most* 2^n *topologically distinct compact spaces of weight* n.

PROOF. Suppose $\{X_\xi : \xi \in \Xi\}$ is a collection of topologically distinct compact spaces of weight n. For each $\xi \in \Xi$, let $\mathscr{B}_\xi$ be a base of cardinality n for X_ξ. For each pair $U, V \in \mathscr{B}_\xi$ with $\bar{U} \cap \bar{V} = \varnothing$, let $f_{U,V}$ be an element of $C(X_\xi)$ such that $f_{U,V}|U=1$ and $f_{U,V}|V=0$. Let $\mathscr{F}_\xi$ denote the subset of $C(X_\xi)$ consisting of all these $f_{U,V}$'s together with all constant functions $f(x)=r$ with r rational. Then card $\mathscr{F}_\xi \leqq n$ and, by the Stone-Weierstrass Theorem, the smallest closed subring of $C(X_\xi)$ containing $\mathscr{F}_\xi$ is $C(X_\xi)$, where $C(X_\xi)$ is given the usual metric.

Now let Z be a fixed discrete space of power n and let $C^*(Z)$ have the usual metric. For each $\xi \in \Xi$, choose $f_\xi : Z \rightarrow X_\xi$ such that $f_\xi[Z]$ is dense in X_ξ, and let $F_\xi : C(X_\xi) \rightarrow C^*(Z)$ be the induced map of f_ξ; i.e., such that $F_\xi(f) = f \circ f_\xi$ for every $f \in C(X_\xi)$. Then each F_ξ is a ring isomorphism so that, for $\xi \neq \xi'$, $F_\xi[C(X_\xi)]$ and $F_{\xi'}[C(X_{\xi'})]$ are nonisomorphic, and hence distinct, subrings of $C^*(Z)$. Furthermore, each F_ξ is an isometry; in particular, since $C(X_\xi)$ is complete, $F_\xi[C(X_\xi)]$ is a closed subspace of

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$C^*(Z)$. Thus the smallest closed subring of $C^*(Z)$ containing $F_\xi[\mathscr{F}_\xi]$ is $F_\xi[C(X_\xi)]$ for every $\xi \in \Xi$. Thus for $\xi \neq \xi'$, $F_\xi[\mathscr{F}_\xi] \neq F_{\xi'}[\mathscr{F}_{\xi'}]$. Consequently, $\{F_\xi[\mathscr{F}_\xi]: \xi \in \Xi\}$ is a collection of distinct subsets of $C^*(Z)$ of cardinality at most n. But $C^*(Z)$ has cardinality 2^n and hence at most 2^n subsets of cardinality at most n. Thus card $\Xi \leqq 2^n$.

PROPOSITION 2. *For every cardinal $m \geqq 2^{\aleph_0}$, there are 2^m topologically distinct continua of power m and weight m.*

PROOF. Let L denote the long line constructed on the set S of all ordinals $\beta \leqq \omega(m)$, the initial ordinal of cardinality m. We regard S as a subset of L. For each $\beta \in S$, let $\langle X_\beta, x_\beta \rangle$ be either $\langle I^2, (0, 0)\rangle$ or $\langle I^3, (0, 0, 0)\rangle$, where $I = [0, 1] \subset R$. Let X be the space obtained by attaching each X_β to L by identifying $x_\beta \in X_\beta$ with $\beta \in S$, and weakening the usual quotient topology by requiring that any neighborhood of a limit ordinal $\gamma \in S$ contains $\bigcup \{X_\beta : \alpha < \beta < \gamma\}$ for some $\alpha < \gamma$. Then X is a continuum of power m and weight m.

Now suppose X' were another such space constructed in the same way but with a conceivably different choice of the $\langle X'_\beta, x'_\beta\rangle$'s, and suppose $f: X \to X'$ were an onto homeomorphism. Let Y denote the set of all points of X at which the dimension is 1; then $L = \overline{Y}$ and $S = \overline{Y} \cap (X - Y)$. Similarly, $L' = \overline{Y}'$ and $S' = \overline{Y}' \cap (X' - Y')$ in X'. It is immediate that $f[Y] = Y'$, and hence $f[L] = L'$ and $f[S] = S'$. But then $f|L$ is monotone, so that $f|S$ is order-preserving and hence the identity. Now consider the subspace $Z = \bigcup \{X_\beta : \beta \in S\}$ of X, which is the disjoint union of the connected subspaces X_β, and the analogous subspace Z' of X'. Because $Z = (X - L) \cup S$ and $Z' = (X' - L') \cup S'$, it follows that $f[Z] = Z'$; in particular, f maps each X_β onto some $X'_{\beta'}$. Because $f|S$ is the identity, it follows that $f[X_\beta] = X'_\beta$ for every $\beta \in S$. Therefore, since I^2 and I^3 are not homeomorphic, it follows that $X'_\beta = X_\beta$ for every $\beta \in S$.

Finally, since there are 2^m different ways to choose the X_β's there are 2^m topologically distinct continua of power m and weight m.

PROPOSITION 3. *For every cardinal $m \geqq 2^{\aleph_0}$, let p be the smallest cardinal for which there is a continuum of power m and weight p. Then for every cardinal n with $p \leqq n \leqq m$, there are 2^n topologically distinct continua of power m and weight n.*

PROOF. Let K be a continuum of power m and weight p. For any cardinal n with $p \leqq n \leqq m$, let L be the long line constructed on the set of all ordinals $\beta \leqq \omega(n)$. Construct X as in Proposition 2; but take X_0 to be $K \times I^2$ and take X_β, for $\beta > 0$, to be either I^2 or I^3. The argument is then similar to Proposition 2.

REMARKS. For the case $m=2^{\aleph_0}$, the requirement that $X_0=K\times I^2$ can be dispensed with. Furthermore, for the case $m=2^{\aleph_0}$, $n=\aleph_0$, all of the constructed continua can be embedded in the plane by choosing X_β to be either I^2 or an annulus. If $m=2^q$ for some q, then p is simply the smallest such q; for then I^p is a continuum of power m and weight p. In particular, if we assume the Generalized Continuum Hypothesis, then the only continua of power m other than those of weight m are (for nonlimit cardinals m) the 2^n topologically distinct continua of weight n where $2^n=m$.

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