PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 40, Number 2, October 1973

AUTOMORPHISMS OF COMMUTATIVE BANACH ALGEBRAS

B. E. JOHNSON

ABSTRACT. This paper presents a new proof of the theorem of Kamowitz and Scheinberg which states that if α is an element of infinite order of the automorphism group of a commutative semi-simple Banach algebra then the spectrum of α contains all complex numbers of absolute value 1. The proof depends on the fact that the only closed translation invariant subalgebras of $l^{\infty}(-\infty, +\infty)$ (pointwise multiplication) for which the restriction of the shift has a complex number of absolute value 1 in its resolvent set are certain spaces of periodic sequences.

Let $\mathfrak{A}$ be a commutative Banach algebra and α an automorphism of $\mathfrak{A}$. In [1] Kamowitz and Scheinberg show that either $\alpha^n=\iota$, the identity map on $\mathfrak{A}$, for some positive integer n, in which case $\sigma(\alpha)$ is a finite union of subgroups of the circle group $\boldsymbol{T}$, or $\boldsymbol{T}\subseteq\sigma(\alpha)$. In this paper we give an entirely different proof of the same result by showing that if E is an open arc in $\boldsymbol{T}$ and $c\in l^{\infty}$ $(=l^{\infty}(\boldsymbol{Z}))$ is such that every element of the translation invariant subalgebra of $l^{\infty}(\boldsymbol{Z})$ generated by c is in $(S-\lambda I)^3 l^{\infty}$ for all λ in E, where S is the translation operator, then c is a periodic sequence. This gives the required theorem by considering the sequences $c_n=\varphi(\alpha^n a)$, $a\in\mathfrak{A}$, φ an element of the spectrum $\Phi_{\mathfrak{A}}$ of $\mathfrak{A}$ and E a component of $\boldsymbol{T}\backslash\sigma(\alpha)$.

If $c\in l^{\infty}$ then $\{(1+n^2)^{-1}c_n\}\in l^1$ so c is the series of Fourier coefficients of a distribution $\hat{c}=(1-D^2)\sum(1+n^2)^{-1}c_n\omega^n$ of order 2 on $\boldsymbol{T}$. We refer the reader to [2], in particular pp. 80–83 for information on distributions. We denote the support of $\hat{c}$ by $\tau(c)$. If $a\in\mathfrak{A}$, $\varphi\in\Phi_{\mathfrak{A}}$ then $\tilde{a}\in l^{\infty}$ is the sequence $\tilde{a}_n=\varphi(\alpha^n a)$.

LEMMA. *If $\lambda_0\in\boldsymbol{T}\backslash\sigma(\alpha)$ then $\lambda_0\in\boldsymbol{T}\backslash\tau(\tilde{a})$ for all $a\in\mathfrak{A}$.*

PROOF. Let $a\in\mathfrak{A}$. We have $\tilde{a}\hat{}=(1-D^2)f$ for some $f\in C(\boldsymbol{T})$. There is a proper open arc E in $\boldsymbol{T}$ containing λ_0 with $E\cap\sigma(\alpha)=\varnothing$. For each λ in E there is b_λ in $\mathfrak{A}$ with $(\alpha-\lambda\iota)^3 b_\lambda=a$ so putting $\tilde{b}_\lambda\hat{}=(1-D^2)g_\lambda$, $g_\lambda\in C(\boldsymbol{T})$

Received by the editors July 17, 1972.
AMS (MOS) subject classifications (1970). Primary 46H05; Secondary 46F10, 47C05.

we get $(1-D^2)f=p_\lambda^3(1-D^2)g_\lambda$ where $p_\lambda(\omega)=\omega-\lambda$ so that on E

$$\begin{aligned} D^2f_0 &= (1-D^2)f = p_\lambda^3 g_\lambda-(2D^2p_\lambda^3)g_\lambda+6D(g_\lambda p_\lambda' p_\lambda^2)-D^2(p_\lambda^3 g_\lambda) \\ &= D^2(p_\lambda^3 h_\lambda) \end{aligned}$$

for some $h_\lambda \in C(E), f_0 \in C(E)$.

Let E' be an interval in $\boldsymbol{R}$ which is mapped one-to-one onto E by $x\mapsto e^{ix}$ and let F_0, H_λ etc. be the functions on E' corresponding to f_0, h_λ etc. Then $D^2F_0=D^2P_\lambda^3H_\lambda$ in the sense of distributions so that there are complex numbers k_λ, l_λ with

$$F_0(y)=P_\lambda^3(y)H_\lambda(y)+k_\lambda y+l_\lambda$$

for all $y\in E'$. Thus if $e^{ix}=\lambda$ we have $F_0(x-h)-2F_0(x)+F_0(x+h)=o(h^2)$ as $h\to 0$ and so the second symmetric derivative of F_0 is 0 at each point of E'. From this it follows [3, p. 23, Theorem 10.7] (I am indebted to Professor T. M. Flett for this reference) that F_0 and $-F_0$ are both convex so that F_0 is linear in E', $D^2f_0=0$ in E and so $\tau(\tilde{a})\subseteq \boldsymbol{T}\backslash E$.

THEOREM. *If $\boldsymbol{T}\backslash\sigma(\alpha)\neq\varnothing$ then $\sigma(\alpha)$ is a finite union of finite subgroups of $\boldsymbol{T}$.*

PROOF. If $\lambda_0\in \boldsymbol{T}\backslash\sigma(a)$ and E is an open arc in $\boldsymbol{T}\backslash\sigma(a)$ containing λ_0 then, by the Lemma, $E\cap\tau(\tilde{a})=\varnothing$ for all a in $\mathfrak{A}$. Let A be the sup norm closure of $\tilde{\mathfrak{A}}$ in l^∞. By the semicontinuity of τ on l^∞, A is a closed translation invariant subalgebra of l^∞ with $E\cap\tau(c)=\varnothing$ for each c in A. Put $T=(\bigcup\{\tau(c);c\in A\})^-$, then $E\cap T=\varnothing$. If $\lambda\in T$, $n\in \boldsymbol{Z}^+$ and J is an open interval in $\boldsymbol{T}$ containing λ^n then we can find an interval I containing λ with $I^n\subset J$, an element c of A with $\tau(c)\cap I\neq\varnothing$ and an element d of l^1 with $\hat{d}\in\mathscr{D}(\boldsymbol{T})$, support $\hat{d}\subset I$ and such that $\hat{d}\hat{c}=(d*c)^\wedge\neq 0$. As A is a closed translation invariant subalgebra, $d*c\in A$, $d*c\neq 0$ so $(d*c)^n\in A$, $\tau[(d*c)^n]\subset J$ and $(d*c)^n\neq 0$. Thus $\lambda^n\in T$.

Let p be an integer greater than 2π (length of $E)^{-1}$. As $T\cap E=\varnothing$ and λ in T implies λ^n is in T, $n=1,2,\cdots$, we see that every element of T is an nth root of unity for some $n\leq p$. Thus T is a subset of the set of $p!$ roots of unity. Let $c\in A$. As in the Lemma, $\hat{c}=(1-D^2)f$ with $f\in C(\boldsymbol{T})$. Because $\tau(c)$ is finite f is of the form $f(\omega)=r\omega+s\bar{\omega}$ on each interval of $\boldsymbol{T}\backslash\tau(c)$, $\hat{c}$ is a combination of δ functions at the points of $\tau(c)$ and so c is a periodic sequence with period dividing $p!$. Thus for all φ in $\Phi_{\mathfrak{A}}$, a in $\mathfrak{A}$ we have $\varphi(\alpha^{p!}a)=\varphi(a)$ which shows $\alpha^{p!}=\iota$ and $\sigma(\alpha)$ consists only of $p!$ roots of unity.

As $\sigma(\alpha)$ is finite each point is an eigenvalue. If $\alpha(a)=\lambda a$, $a\neq 0$ then $\alpha(a^n)=\lambda^n a^n$ where, as $\mathfrak{A}$ is semisimple, $a^n\neq 0$ so that if $\lambda\in\sigma(\alpha)$, $n\in\boldsymbol{Z}^+$ then $\lambda^n\in\sigma(\alpha)$. It follows from this that if $\sigma(\alpha)$ contains one primitive

mth root of unity it contains all mth roots of unity and so is a finite union of finite subgroups of $\boldsymbol{T}$.

The following result, due to Singer and Wermer, is a corollary of the theorem of Kamowitz and Scheinberg.

COROLLARY. *Let D be a continuous derivation on the commutative Banach algebra $\mathfrak{B}$. Then $D\mathfrak{B} \subseteq$ radical of $\mathfrak{B}$.*

PROOF. Replacing D by tD if necessary $(0<t<1)$, we can assume $\|D\|<\frac{1}{2}$. Then $\beta=e^D$ is an automorphism of $\mathfrak{B}$ with $\|\iota-\beta\|<e^{1/2}-1<1$. Put $\mathfrak{A}=\mathfrak{B}/\text{rad } \mathfrak{B}$. Because rad $\mathfrak{B}$ is invariant under β, β induces an automorphism α of $\mathfrak{A}$ with $\|\iota-\alpha\|<1$ and hence $\sigma(\alpha)\subseteq\{z : z \in \boldsymbol{C},\ |z-1|<1\}$. Thus in the second last paragraph of the proof of the theorem we have length of $E>\pi$ and can take $p=1$, giving $\alpha=\iota$. Thus $\iota-\beta$ maps $\mathfrak{B}$ into rad $\mathfrak{B}$ and hence so does $D=\log \beta=-\sum n^{-1}(\iota-\beta)^n$.

REFERENCES

1. H. Kamowitz and S. Scheinberg, *The spectrum of automorphisms of Banach algebras*, J. Functional Analysis **4** (1969), 268–276. MR **40** #3316.

2. L. Schwartz, *Théorie des distributions*, Tome II, Actualités Sci. Indust., 1122, Hermann, Paris, 1951. MR **12,** 833.

3. A. Zygmund, *Trigonometrical series*. Vol. I, 2nd ed. reprinted with corrections and some additions, Cambridge Univ. Press, New York, 1968. MR **38** #4882.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEWCASTLE UPON TYNE, NEWCASTLE UPON TYNE, ENGLAND