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A GENERALIZATION OF THE BANACH-STONE THEOREM[1]

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ABSTRACT. In this paper the following generalization of the Banach-Stone theorem is proved: If ϕ is a linear isomorphism of an extremely regular linear subspace of $C_0(X)$ onto such a subspace of $C_0(Y)$ with $\|\phi\|\ \|\phi^{-1}\|<2$ then X and Y are homeomorphic.

If X is a locally compact space,[3] we denote by $C_0(X)$ ($C_0^r(X)$) the Banach space of continuous, complex- (real-) valued functions vanishing at infinity on X, provided with the usual supremum norm. (We recall that if X is actually compact, $C_0(X)$ ($C_0^r(X)$) coincides with the set of all continuous, complex- (real-) valued functions on X and is denoted by $C(X)$ ($C^r(X)$).) We call a closed linear subspace A of $C_0(X)$ *completely regular* (*extremely regular*) if for each $x \in X$, each open neighborhood V of x (and each real number ε with $0<\varepsilon<1$) there is a function $f \in A$ such that $1=\|f\|=f(x)>\sup\{|f(x')|: x' \in X\backslash V\}$ ($1=\|f\|=f(x)>\varepsilon\geqq|f(x')|$ for every $x' \in X\backslash V$). Clearly, every extremely regular function space is completely regular. But the converse is false. In [**3**] it is shown that if X has at least three points then $C_0(X)$ has proper completely regular linear subspaces while $C_0(X)$ has a proper extremely regular linear subspace if, and only if, X is nondispersed, that is, the αth derived set $X^{(\alpha)}$ of X is nonvoid for every ordinal number α.

The well-known Banach-Stone theorem states that if $C_0(X)$ and $C_0(Y)$ are isometrically isomorphic then X and Y are homeomorphic.

Myers [**4**] has proved that a sufficient condition for compact spaces X and Y to be homeomorphic is that a completely regular linear subspace of $C^r(X)$ and such a subspace of $C^r(Y)$ be isometrically isomorphic.

Cambern [**2**] has shown that if there is a linear isomorphism ϕ of $C_0(X)$ onto $C_0(Y)$, for any locally compact spaces X and Y, such that $\|\phi\|\ \|\phi^{-1}\|<2$, then X and Y are homeomorphic. (Amir [**1**] proved this result, independently, in the special case that X and Y are compact and ϕ is from $C^r(X)$ onto $C^r(Y)$.)

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[3] Throughout this paper all topological spaces are assumed to be Hausdorff.

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The purpose of this article is to prove the following theorem, which, in certain special cases, combines Myers' and Cambern's results.

THEOREM. *Let X and Y be locally compact spaces and let A and B be extremely regular linear subspaces of $C_0(X)$ and $C_0(Y)$ respectively. If there is a linear isomorphism ϕ of A onto B with $\|\phi\|\ \|\phi^{-1}\|<2$, then X and Y are homeomorphic.*

Before beginning the proof of the theorem, we establish some conventions regarding notation. Let μ be a finite regular Borel measure on the locally compact space X. For a Borel set F in X, $\mu/F(f)$, $f\in C_0(X)$, denotes the μ-integral of f over F. If $F=X$ we use $\mu(f)$ instead of $\mu/X(f)$. $\|\mu\|$ denotes $|\mu|(X)$, where $|\mu|$ is the total variation of μ. For a point $x\in X$, μ_x denotes the unit point mass at x.

Let X, Y, A, B and ϕ be as in the theorem. We may assume that ϕ is norm-increasing and that $\|\phi^{-1}\|=1$. (If not, we take $\psi=\|\phi^{-1}\|\phi$ which has these properties.) For each $y\in Y$ ($x\in X$), $\phi^*\mu_y$ ($\phi^{*-1}\mu_x$) will denote the linear functional on A (on B) defined by: $\phi^*\mu_y(f)=\phi(f)(y)$ ($\phi^{*-1}\mu_x(g)=\phi^{-1}(g)(x)$) for each $f\in A$ ($g\in B$), where ϕ^* denotes the adjoint of ϕ. For each $y\in Y$ ($x\in X$), we fix a finite regular Borel measure $\nu(y)$ on X ($\nu(x)$ on Y) such that $\nu(y)(f)=\phi^*\mu_y(f)$ ($\nu(x)(g)=\phi^{*-1}\mu_x(g)$) for each $f\in A$ ($g\in B$) and $\|\nu(y)\|=\|\phi^*\mu_y\|$ ($\|\nu(x)\|=\|\phi^{*-1}\mu_x\|$).

If $\{U_i:i\in I\}$ denotes the family of all open neighborhoods of a point $x\in X$, the index set I will always be assumed partially ordered by the relation that $i\geqq j$ if, and only if, $U_i\subset U_j$.

In the proof of the theorem we shall employ the techniques of **[2]**. Let M be a positive number such that $\|\phi\|<2M<2$, and let $M'=\|\phi\|-M$, $N=\frac{1}{2}M$ and $N'=1-N$. It follows that for each $x\in X$ there exists at most one $y\in Y$ such that $|\nu(x)(\{y\})|>N$, and for each $y\in Y$ there is at most one $x\in X$ such that $|\nu(y)(\{x\})|>M$. Define $Y_1=\{y\in Y:|\nu(y)(\{x\})|>M$ for some $x\in X\}$ and $X_1=\{x\in X:|\nu(x)(\{y\})|>N$ for some $y\in Y\}$. Now let us define ρ from Y_1 into X by: $\rho(y)=x$ such that $|\nu(y)(\{x\})|>M$, and similarly define τ from X_1 into Y by: $\tau(x)=y$ such that $|\nu(x)(\{y\})|>N$.

PROOF OF THE THEOREM. We shall fix a decreasing sequence ε_n of positive numbers with $0<\varepsilon_n<1$ and $\lim_n \varepsilon_n=0$. If $\{V_i:i\in I\}$ is the set of all open neighborhoods of $x\in X$, we call a family of functions $\{f_{in}:i\in I;\ n=1,2,\cdots\}\subset C_0(X)$ *an extremely regular system* at x if $1=\|f_{in}\|=f_{in}(x)>\varepsilon_n\geqq|f_{in}(z)|$ for every $z\in X\setminus V_i$.

We shall need the following simple lemmas.

LEMMA 1. *Let μ be a finite regular Borel measure on X, x be a point in X and $\{V_i:i\in I\}$ be the set of all open neighborhoods of x. For each $i\in I$, let f_i be a measurable function on X vanishing outside V_i with $1=f_i(x)=\sup\{|f_i(z)|:z\in X\}$. Then, $\lim_i \mu(f_i)=\mu(\{x\})$.*

LEMMA 2. *Let $x_0 \in X$ (resp. $y_0 \in Y$) be arbitrary, and let $\{f_{in}: i \in I; n=1, 2, \cdots\} \subset A$ (resp. $\{g_{jn}: j \in J; n=1, 2, \cdots\} \subset B$) be an extremely regular system at x_0 (resp. at y_0). Now let $Y(x_0, n)$ (resp. $X(y_0, n)$) denote the set of all $y \in Y$ (resp. $x \in X$) such that y (resp. x) is a cluster point of a net $\{y_i: i \in I\}$ (resp. $\{x_j: j \in J\}$) with $|\phi(f_{in})(y_i)| > M$ (resp. $|\phi^{-1}(g_{jn})(x_j)| > N$). Then, $Y(x_0, n)$ (resp. $X(y_0, n)$) is finite for every n, provided $\varepsilon_n < M/\|\phi\| - \frac{1}{2}$ (resp. $\varepsilon_n < \frac{1}{2}M - \frac{1}{2}$).*

PROOF. As in the proof Proposition 1 of [**2**, p. 1063] one can easily show that if $y \in Y(x_0, n)$, $|\phi^{-1}(g)(x_0)| \geqq M/\|\phi\| - \frac{1}{2}$ for each $g \in B$ with $M = \|g\| = g(y)$. Let $y_1, y_2, \cdots, y_m$ be m distinct points of $Y(x_0, n)$ and let $U_1, U_2, \cdots, U_m$ be disjoint open sets with $y_j \in U_j$. For each $1 \leqq k \leqq m$ let $g_k \in B$ such that $M = \|g_k\| = g_k(y_k)$ and $|g_k(y)| \leqq M/m$ for every $y \in Y \setminus U_k$. Choose complex numbers $\lambda_1, \lambda_2, \cdots, \lambda_m$ with $|\lambda_k| = 1$ and such that $\lambda_1\phi^{-1}(g_1)(x_0), \cdots, \lambda_m\phi^{-1}(g_m)(x_0)$ have equal arguments. Then we get:

$$2M > \left\| \sum_{k=1}^{m} \lambda_k g_k \right\| \geqq \left\| \phi^{-1}\left(\sum_{k=1}^{m} \lambda_k g_k \right) \right\|$$
$$\geqq \left| \sum_{k=1}^{m} \lambda_k \phi^{-1}(g_k)(x_0) \right| \geqq m(M/\|\phi\| - \tfrac{1}{2}),$$

from which we conclude that $Y(x_0, n)$ is finite.

To prove that $X(y_0, n)$ is finite, replace ϕ by $\psi = \|\phi\|\phi^{-1}$, M by $M_0 = \|\phi\|N$ and interchange X and Y in the above proof.

LEMMA 3. *Let x and y be arbitrary points of X and Y respectively and $\{g_{jn}: j \in J; n=1, 2, \cdots\} \subset B$ be an extremely regular system at y. If $N_1 > N$ and $\varepsilon_n < N_1 - N$ then the inequality $|\nu(x)(\{y\})| > N_1$ implies that x is a cluster point of a net $\{x_j: |\phi^{-1}(g_{jn})(x_j)| > N; j \in J\}$.*

PROOF. Let λ be any element of the cluster set of the net $\{|\phi^{-1}(g_{jn})(x)|: j \in J\}$. It is easy to show that $\lambda \geqq |\nu(x)(\{y\})| - \varepsilon_n > N_1 - (N_1 - N) = N$. (Use Lemma 1 and the fact that λ is the limit of a subnet of $\{|\nu(x)(g_{jn})|: j \in J\}$.) Now it is clear that x is a cluster point of the net $\{x_j: j \in J\}$, where $x_j = x$ if $|\phi^{-1}(g_{jn})(x)| > N$ and x_j is any point such that $|\phi^{-1}(g_{jn})(x_j)| > N$, otherwise.

As in [**2**] the proof of the theorem is now completed by means of three propositions.

PROPOSITION 1. *ρ (resp. τ) is a mapping of Y_1 (resp. X_1) onto X (resp. Y).*

PROOF. Let x be any point of X. We want to show that there exists $y \in Y$ such that $|\nu(y)(\{x\})| > M$. To this end, we fix a real number M_1 with $M < M_1 < 1$ and assume that $\varepsilon_1 < \min\{(M_1 - M)/4,\ M/\|\phi\| - \frac{1}{2}\}$. Let $\{V_i: i \in I\}$ denote the set of all open neighborhoods of x and let $\{f_{in}: i \in I; n=1, 2, \cdots\} \subset A$ be an extremely regular system at x. Since $\varepsilon_1 < M/\|\phi\| - \frac{1}{2}$,

$Y(x, 1)$ is finite (Lemma 2). Suppose that $|\nu(y)(\{x\})| \leqq M$ for each $y \in Y(x, 1)$. Then from Lemma 1 and the inequality $|\nu(y)(f_{i1})| \leqq |[\nu(y)/V_i(f_{i1})]| + 2\varepsilon_1$ for each $y \in Y$ and each $i \in I$, it follows that there exists $i_1 \in I$ such that $|\nu(y)(f_{i1})| < M + 4\varepsilon_1 < M_1$ for all $i \geqq i_1$ and all $y \in Y(x, 1)$.

Let $\mu = \nu(x) - \sum_{y \in Y(x,1)} \beta_y \mu_y$, where $\beta_y = \nu(x)(\{y\})$. Since μ is identically zero on $Y(x, 1)$ there exists a compact K in $Y \setminus Y(x, 1)$ such that $|\mu|(Y \setminus K) < (1 - M_1)/4$. Since K is compact and disjoint from $Y(x, 1)$ there exists $i_2 \in I$ such that $|\phi(f_{in})(y)| \leqq M$ for all $i \geqq i_2$ and all $y \in K$. For each $i \in I$, $i \geqq i_1$ and $i \geqq i_2$ we have (noting that $\sum_{y \in Y(x,1)} |\beta_y| \leqq 1$ and $\|\mu\| \leqq 1 - \sum_{y \in Y(x,1)} |\beta_y|$)

$$\begin{aligned} 1 &= f_{i1}(x) = \nu(x)(\phi(f_{i1})) \\ &= \sum_{y \in Y(x,1)} \beta_y \mu_y(\phi(f_{i1})) + \mu/K(\phi(f_{i1})) + \mu/(Y \setminus K)(\phi(f_{i1})) \\ &< \sum_{y \in Y(x,1)} |\beta_y| M_1 + M \|\mu\| + 2(1 - M_1)/4 \\ &< \sum_{y \in Y(x,1)} |\beta_y| M_1 + M\left(1 - \sum_{y \in Y(x,1)} |\beta_y|\right) + (1 - M_1)/2 < 1, \end{aligned}$$

which is absurd.

To prove that τ maps X_1 onto Y we let $\psi = \|\phi\| \phi^{-1}$ and $M_0 = \|\phi\| N$. By the above discussion, for each $y \in Y$ there exists $x \in X$ such that $|\gamma(x)(\{y\})| > M_0$, where $\gamma(x) = \|\phi\| \nu(x)$. Hence $|\nu(x)(\{y\})| > N$.

PROPOSITION 2. *If $y \in Y_1$, $\rho(y) = x$, then $x \in X_1$ and $\tau(x) = y$.*

PROOF. Let $y \in Y_1$, $x = \rho(y)$ and let $\{g_{jn} : j \in J;\ n = 1, 2, \cdots\} \subset B$ be an extremely regular system at y. We know that if either $x \notin X_1$ or $x \in X_1$ but $\tau(x) \neq y$ then $|\nu(x)(\{y\})| \leqq N$. Suppose that $|\nu(x)(\{y\})| \leqq N$. Choose a positive number N_1 such that $N < N_1 < 1/\|\phi\|$. Then $P > N_1$, where $P = \sup\{|\nu(x')(\{y\})| : x' \in X\}$. (For if we call $M_1 = \frac{1}{2} N_1$, then $\|\phi\| < 2M_1 < 2$, therefore, by Proposition 1 there exists $x' \in X$ with $|\nu(x')(\{y\})| > N_1$.) Let ε be a positive number less than $N_1 - N$ and $(P - \varepsilon)M > (P + \varepsilon)M' + \varepsilon$. We may assume that $\varepsilon_1 < \varepsilon/2$ and $\varepsilon_1 < \frac{1}{2}M - \frac{1}{2}$. Then, by Lemma 2, $X(y, 1)$ is finite, and by Lemma 3, $|\nu(x')(\{y\})| > N_1$ implies that $x' \in X(y, 1)$. Thus, $P = \max\{|\nu(x')(\{y\})| : x' \in X(y, 1)\} = |\nu(x_1)(\{y\})|$ for some $x_1 \in X(y, 1)$. It is easy to show (by Lemma 1) that there exists $j_1 \in J$ such that $P - \varepsilon < |\nu(x_1)(g_{j1})|$ and $|\nu(x')(g_{j1})| < P + \varepsilon$ for all $j \geqq j_1$, all $x' \in X(y, 1)$ and $x' \neq x_1$. Since $\tau(x_1) = y$ and $x_1 \neq x$, there exists $y_1 \in Y$, $y_1 \neq y$ and $\rho(y_1) = x_1$. Let $\nu = \nu(y_1) - \sum_{x' \in X(y,1)} \alpha_{x'} \mu_{x'}$ where $\alpha_{x'} = \nu(y_1)(\{x'\})$. Since $|\alpha_{x_1}| > M$, $M' > \sum_{x' \in X(y,1)} |\alpha_{x'}| + \|\nu\| - |\alpha_{x_1}|$. Since ν is identically zero on $X(y, 1)$ there exist a compact set $K \subset X \setminus X(y, 1)$ and $j_2 \in J$ such that $|\nu|(X \setminus K) \leqq (P + \varepsilon - N)\|\nu\|$ and $|\phi^{-1}(g_{j1})(x')| \leqq N$ for all $j \geqq j_2$ and all $x' \in K$. Let $j_0 \in J$

such that $j_0 \geqq j_1$, $j_0 \geqq j_2$ and $y_1 \notin V_{j_0}$. Then we obtain:

$$\nu(y_1)(h) = \alpha_{x_1} h(x_1) + \sum_{x' \in X(y,1): x' \neq x_1} \alpha_{x'} h(x') + \nu/K(h) + \nu/(X \backslash K)(h),$$

where $h = \phi^{-1}(g_{j_0 1})$. $|\nu(y_1)(h)| < \varepsilon$, $|\alpha_{x_1} h(x_1)| > M(P-\varepsilon)$ and the modulus of the sum of the remaining terms of the right-hand side is less than $(P+\varepsilon)M'$. Thus, we get $(P-\varepsilon)M < (P+\varepsilon)M' + \varepsilon$, which contadicts the choice of ε. Hence, $x \in X_1$ and $\tau(x) = y$, proving the proposition.

PROPOSITION 3. *τ is a closed map of X onto Y.*

PROOF. Let F be a closed subset of X. Let y be any point in $Y \backslash \tau(F)$ and let $x = \rho(y)$. Now let $\{f_{in} : i \in I;\ n = 1, 2, \cdots\} \subset A$ be an extremely regular system at x. We may assume that $\varepsilon_1 < \min\{(M-M')/2, 2(|\alpha| - M)/3\}$, where $\alpha = \nu(y)(\{x\})$. Choose $i_0 \in I$ such that $|\phi(f_{i_0 1})(y)| > |\alpha| - \varepsilon_1 > M$. It is easy to see that $|\phi(f_{i_0 1})(y')| \leqq 2\varepsilon_1 + M' < M$ for each $y' \in \tau(F)$. Thus, for each $x \in X \backslash F$ there exists $f_x \in A$ such that

(i) $|\phi(f_x)(y')| < M$ for all $y' \in \tau(F)$,

(ii) $|\phi(f_x)(y)| > M$ where $y = \tau(x)$.

Now define $E_x = \{y \in Y : |\phi(f_x)(y)| \leqq M\}$, $x \in X \backslash F$. Each E_x is a closed set. Thus $\bigcap_{x \in X/F} E_x = \tau(F)$ is closed in Y. Thus, τ is a closed map, which implies that ρ is continuous. Similarly $\tau = \rho^{-1}$ is continuous. This completes the proof of the theorem.

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