PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 41, Number 1, November 1973

SHORTER NOTES

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A MEASURE-THEORETIC PROOF OF A THEOREM ON REFLEXIVITY[1]

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ABSTRACT. Representing measures are employed to give a short proof of the following theorem: *A normed linear space is reflexive if its unit ball is weakly compact.*

We give here a short proof of a well-known theorem on reflexive Banach spaces. The reader familiar with the theory of functions algebras, say, will recognize the use of the Hahn-Banach and Riesz-Kakutani theorems to obtain "representing measures." The reader familiar with the theory of measures on compact convex sets will note that the last half of this proof is just the construction of a resolvent for λ.

THEOREM. *Let B be a normed linear space. If $\Sigma(B)$, the unit ball of B, is compact in its weak topology, then B is reflexive.*

PROOF. Let $\psi \in B^{**}$ have $\|\psi\| \leqq 1$. By restriction B^* is norm isomorphic to a subspace of $C(U)$ where U is the compact Hausdorff space obtained by giving $\Sigma(B)$ its weak topology. Hence, by the Hahn-Banach and Riesz-Kakutani theorems, there exists a (not necessarily positive) Borel measure λ on U, of total variation $\leqq 1$, such that $\psi(f)=\int_U f\,d\lambda$ for every $f \in B^*$. This λ is the limit in the weak-star topology of $C(U)^*$ of a net of measures $\{\lambda_\alpha\}_{\alpha\in A}$, where each λ_α is of the form

$$\sum_{i=1}^{n_\alpha} \lambda(E_i^\alpha)\, \delta_{x_i^\alpha}$$

Received by the editors April 6, 1973.

AMS (MOS) subject classifications (1970). Primary 46B10; Secondary 46A25, 46E15.

Key words and phrases. Reflexive normed linear space, weak topologies, weak-star topologies, representing measure.

[1] This research was supported by an NDEA Title IV graduate fellowship.

for some finite Borel partition $\{E_i^\alpha\}_{i=1}^{n_\alpha}$ of U and where $x_i^\alpha \in E_i^\alpha$. Since the total variation of λ is $\leqq 1$, $\sum_{i=1}^{n_\alpha} \lambda(E_i^\alpha)x_i^\alpha \in U$ for each α. If $x \in U$ is a weak limit point of some subnet of the net $\{\sum_{i=1}^{n_\alpha} \lambda(E_i^\alpha)x_i^\alpha\}_{\alpha\in A}$, then $\psi(f)= \int_U f\, d\lambda = f(x)$ for each $f \in B^*$.

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