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NOTE ON A LIMIT-POINT CRITERION

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ABSTRACT. A sufficient condition is given for the formal differential operator $\tau y(t)=(p(t)y'(t))'+q(t)y(t)$ defined on the interval $[a, b)$, $b\leqq\infty$, to be of limit-point type at b; this generalizes a criterion of Ismagilov given for the case $p(t)=1$ and $b=\infty$.

We consider the linear second order differential operator τ defined on the real half-open interval $[a, b)$, $b\leqq\infty$, by

$$\tau y(t) = (p(t)y'(t))' + q(t)y(t) \tag{1}$$

where $p^{-1}(t)>0$ and $q(t)$ are real-valued functions locally Lebesgue integrable on $[a, b)$. The operator τ is said to be of limit-circle type at b if every solution $f(t)$ of the differential equation $\tau y(t)=0$ satisfies the condition

$$\int_a^b |f^2(t)|\, dt < \infty. \tag{2}$$

If this is not the case, then τ is said to be of limit-point type at b.

In [1], Ismagilov proved that if $p(t)=1$ and $q(t)\leqq -q_n<0$ on disjoint intervals $I_n\subset[a,\infty)$ of length μ_n and $\sum_{n=1}^{\infty} q_n^{1/2}\mu_n^3=\infty$ then τ is of limit-point type at ∞. We present here the corresponding result for the general operator τ defined on an arbitrary half-open subinterval of the real line.

THEOREM. *If there exist finite disjoint subintervals I_n, $n=1, 2, \cdots$, of $[a, b)$ and corresponding numbers q_n and p_n such that $q(t)\leqq -q_n<0$ and $p(t)\geqq p_n>0$ on I_n and*

$$\sum_{n=1}^{\infty} p_n^{3/2}q_n^{1/2}\left(\int_{I_n} p^{-1}(s)\, ds\right)^3 = \infty \tag{3}$$

then the operator τ defined by equation (1) *is of limit-point type at b.*

PROOF. For each $n=1, 2, \cdots$, let the interval I_n have endpoints a_n and b_n, where $a_n<b_n$, and let $P(s)=\int_a^s p^{-1}(u)\, du$ for any $s\in[a, b)$. We define the function $h(s)$ on $\bigcup_n I_n$ by

$$h(s) = (P(b_n) - P(a_n))^{-2}(P(s) - P(a_n))^2(P(b_n) - P(s))^2 \tag{4}$$

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for $s \in I_n$. Then $h(a_n)=h(b_n)=h'(a_n)=h'(b_n)=0$ and, on each interval I_n,

$$
\begin{aligned}
& p(s)(p(s)h'(s))' \\
(5) \qquad & = 2(P(b_n) - P(a_n))^{-2}[6P^2(s) - 6P(s)(P(b_n) + P(a_n)) \\
& \qquad + P^2(a_n) + 4P(a_n)P(b_n) + P^2(b_n)] \leqq 2
\end{aligned}
$$

because this is a quadratic function in the variable $P(s)$ which takes its maximum value at either endpoint of I_n. Also, after integrating by parts twice, it is easily seen that for any solution $f(s)$ of $\tau y(s)=0$,

$$
(6) \qquad \int_{I_n} f^2(s)(p(s)h'(s))'\,ds = 2\int_{I_n} h(s)[p(s)(f'(s))^2 - q(s)f^2(s)]\,ds.
$$

Now let $u(s)$ and $v(s)$ be any two independent real-valued solutions of $\tau y(s)=0$ for which $p(uv'-u'v)=1$. Then, on each interval I_n,

$$
(7) \qquad \begin{aligned}
2(-q)^{1/2}p^{-1/2} &= 2(-q)^{1/2}u(p^{1/2}v') - 2(-q)^{1/2}v(p^{1/2}u') \\
&\leqq p[(u')^2 + (v')^2] - q[u^2 + v^2].
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int_a^b (u^2 + v^2)\,ds &\geqq \sum_{n=1}^{\infty} \int_{I_n} (u^2 + v^2)\,ds \\
&\geqq \sum_{n=1}^{\infty} \left[\sup_{I_n} p(ph')'\right]^{-1} \int_{I_n} (u^2 + v^2)p(ph')'\,ds \\
&\geqq \sum_{n=1}^{\infty} p_n\left[\sup_{I_n} p(ph')'\right]^{-1} \int_{I_n} (u^2 + v^2)(ph')'\,ds \\
&= 2\sum_{n=1}^{\infty} p_n\left[\sup_{I_n} p(ph')'\right]^{-1} \\
&\qquad \times \int_{I_n} h[p((u')^2 + (v')^2) - q(u^2 + v^2)]\,ds \quad \text{from (6)} \\
(8) \qquad &\geqq 4\sum_{n=1}^{\infty} p_n\left[\sup_{I_n} p(ph')'\right]^{-1} \int_{I_n} h(-q)^{1/2}p^{-1/2}\,ds \quad \text{from (7)} \\
&\geqq 4\sum_{n=1}^{\infty} q_n^{1/2}p_n^{3/2}\left[\sup_{I_n} p(ph')'\right]^{-1} \int_{I_n} h(s)p^{-1}(s)\,ds \\
&\geqq 2\sum_{n=1}^{\infty} q_n^{1/2}p_n^{3/2} \int_{I_n} h(s)p^{-1}(s)\,ds \quad \text{using the property (5)} \\
(9) \qquad &= \frac{1}{15}\sum_{n=1}^{\infty} q_n^{1/2}p_n^{3/2}\left(\int_{I_n} p^{-1}(s)\,ds\right)^3 = \infty.
\end{aligned}
$$

This completes the proof. Q.E.D.

At this stage, one might reasonably expect to improve this theorem by using another function $g(s)$ satisfying

$$g(a_n) = g(b_n) = g'(a_n) = g'(b_n) = 0, \qquad n = 1, 2, \cdots,$$

in the proof, instead of the function $h(s)$ defined by (4). Unfortunately, no such improvement is possible. To see this, let

$$T_n(g) = \left[\sup_{I_n} p(pg')'\right]^{-1} \int_{I_n} g(s)p^{-1}(s)\, ds,$$

and substitute $g(s)$ for $h(s)$ in the preceding proof; the inequality (8) can now be rewritten in the form

$$\int_a^b (u^2 + v^2)\, ds \geqq 4 \sum_{n=1}^{\infty} q_n^{1/2} p_n^{3/2} T_n(g)$$

which shows that τ is of limit-point type at b if we have $\sum_{n=1}^{\infty} q_n^{1/2} p_n^{3/2} T_n(g) = \infty$. But,

$$T_n(g) = \frac{1}{2}\left[\sup_{I_n} p(pg')'\right]^{-1} \int_{I_n} (P(s) - P(a_n))^2 (p(s)g'(s))'\, ds$$

$$\leqq \frac{1}{2} \int_{I_n} (P(s) - P(a_n))^2 p^{-1}(s)\, ds = \frac{1}{6}\left(\int_{I_n} p^{-1}(s)\, ds\right)^3$$

and this criterion is therefore only a special case of the theorem. Thus, choosing another function $g(s)$ can at best only improve the constant appearing in the inequality (9).

Reference

1. R. S. Ismagilov, *On the self-adjointness of the Sturm-Liouville operator*, Uspehi Mat. Nauk **18** (1963), no. 5 (113), 161–166. (Russian) MR **27** #4979.

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