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BOREL'S FIXED POINT THEOREM FOR KAEHLER MANIFOLDS AND AN APPLICATION

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ABSTRACT. A short proof of a generalization of the Borel fixed point theorem to the case of Kaehler manifolds is given and, as an application, a short proof of Wang's theorem that compact simply connected homogeneous manifolds are projective and of the form G/P, where G is a complex semisimple Lie group and P is a parabolic subgroup.

I will give a short proof of a generalization of the Borel fixed point theorem to the case of Kaehler manifolds and, as an application, give a short proof of Wang's theorem that compact simply connected homogeneous Kaehler manifolds are projective and of the form G/P, where G is a complex semisimple Lie group and P is a parabolic subgroup.

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PROPOSITION I. *Let X be a compact Kaehler manifold with $H^1(X, C)=0$, and let S be a solvable connected complex Lie group acting holomorphically on X. Let Y be a subvariety of X invariant under S. Then S has a fixed point on Y and the fixed points form a subvariety.*

PROOF. First assume Y is a manifold, S is one dimensional and has no fixed points on Y. Associated to S we have a holomorphic tangent field on X and by invariance of Y under S, also on Y; call it A. By assumption A has no zeroes on Y.

We have short exact sequences where $\mathcal{O}_Y$, $\mathcal{O}_X$ are the holomorphic structure sheaves and Ω^1_X, Ω^1_Y are the holomorphic one forms and M is a subsheaf of $\mathcal{O}_X$. A and $A|_Y$ as sections of the dual sheaves of Ω^1_X and

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Ω_Y^1 respectively give rise to natural maps denoted by the same letters. r stands for the restriction map from X to Y.

$$\begin{tikzcd}
0 \arrow[r] & F^1 \arrow[r] \arrow[d] & \Omega_X^1 \arrow[r, "A"] \arrow[d, "r"] & M \arrow[r] \arrow[d, "r"] \arrow[dr] & 0 \\
 & & & & \mathcal{O}_X \arrow[dl, "r"] \\
0 \arrow[r] & F^2 \arrow[r] & \Omega_Y^1 \arrow[r, "A|_Y"] & \mathcal{O}_Y \arrow[r] & 0
\end{tikzcd}$$

A having no zeroes is equivalent to $\Omega_Y^1 \xrightarrow{A|_Y} \mathcal{O}_Y$ being a surjection, and F^2 being locally free of rank $n-1$.

Passing to cohomology we have the commutative diagram:

$$\begin{tikzcd}
H^1(X, \Omega_X^1) \arrow[r] \arrow[d] & H^1(X, M) \arrow[d] \arrow[dr] & \\
 & & H^1(X, \mathcal{O}_X) \arrow[dl] \\
H^1(Y, \Omega_Y^1) \arrow[r] & H^1(Y, \mathcal{O}_Y) &
\end{tikzcd}$$

The Kaehler form ω on X restricts to a Kaehler form $\omega|_Y$ on Y. Since $H^1(X, \mathcal{O}_X)$ is a subgroup of $H^1(X, C)$ on a Kaehler manifold, it equals zero. Thus the image of $\omega|_Y$ in $H^1(Y, \mathcal{O}_Y)$ is 0 and thus the Kaehler form of Y is an element of $H^1(Y, F^2)$. By the Dolbeault isomorphism we can represent $\omega|_Y$ by a F^2 valued 0, 1 form. I observe that the nth exterior power of $\omega|_Y$, where the dimension of $Y=n$, would be zero, since the fibre dimension of $F^2=n-1$. On the other hand the nth power of $\omega|_Y$ is a nontrivial element of $H^{2n}(Y, C)$, a volume form, which gives a contradiction.

We now remove the assumption that Y is nonsingular. Simply note that Y has a finite filtration by singular varieties

$$Y = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_m,$$

where Y_s is the singular set of Y_{s-1}. This filtration is respected by S and the last element of the filtration is a manifold which we can take as Y.

The fixed point set is the zero set of A and so is a subvariety.

Now let S be of dimension n and assume the proposition is true for $n-1$. Since S is solvable it has a normal subgroup S' of dimension $n-1$. By assumption S' has a nontrivial fixed point variety Y' in Y. Note that S leaves this new Y' invariant. To see this, take element s of S, and an element y of Y' and note that $S'(sy)=s(S'y)=sy$, so sy is a fixed point of S' in Y, that is, an element of Y'.

Pick an A belonging to the complex Lie algebra of S and not belonging to the Lie subalgebra of S'. A gives rise to a one parameter subgroup of T the universal cover of S, and hence to a tangent vector field on Y' and the fixed point set of S in Y is the fixed point set of T in Y', and the first part of the proof applies. Q.E.D.

PROPOSITION II. *Let X be a compact homogeneous Kaehler manifold with $H^1(X, C)=0$. Then X is a projective manifold of the form G/P where G, the connected component of the identity of X's complex Lie group of biholomorphic transformations is semisimple, and P is parabolic, that is, contains a maximal solvable connected subgroup.*

PROOF. If G were not semisimple it would have a solvable radical N. Let $x \in X$ be a fixed point of N which exists by Proposition I. For all $g \in G$ we have $N(gx)=g(Nx)=gx$. But G is transitive and so N leaves every point fixed and is thus the identity.

Let B be a maximal solvable connected subgroup of G. B has a fixed point x and so B belongs to the stabilizer of x, which we call P. Thus x is of the form G/P where P is parabolic. Q.E.D.

REMARKS. It is easy to give simply connected complex manifolds where the Borel fixed point theorem is false, e.g. the Calabi-Eckmann manifolds.

The correct setting is probably on an appropriate generalization of Kaehler manifolds that one might call pseudo-Kaehler manifolds. These would be complex compact manifolds with a closed C^∞ 1, 1 form which is positive definite on a Zariski open set. Hodge theory should go through for these manifolds, and then the above proof will be applicable.

The general structure of a compact Kaehlerian homogeneous space is given in **[1]**.

The proof of the fixed point theorem above, looked at in the context of the Albanese map, actually proves a strong converse.

PROPOSITION. *Let S be a complex connected solvable Lie group acting holomorphically on a compact Kaehler manifold X. S has a fixed point on any subvariety, including X, that S leaves invariant if and only if the complex Lie algebra of holomorphic vector fields on X associated to S is annihilated by every holomorphic one form.*

If S has a fixed point, then the fixed point subvariety surjects onto the image of X in its Albanese variety under the Albanese map.

In the case that there are no holomorphic one forms, that is $H^1(X, C)=0$, the above result reduces to the theorem in the paper.

A corollary of the above proposition is that a compact Kaehler manifold that possesses a nowhere vanishing holomorphic vector field A, also possesses a holomorphic one form a such that $a(A)\neq 0$.

This proof and other related results will be presented in a future article.

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