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# COVERING DIMENSION IN FINITE-DIMENSIONAL METRIC SPACES

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ABSTRACT. Let $P:2^V \to 2^V$ be a structure in a topological space $V$ such that $P(\varnothing)=\varnothing$, $P(\{x\})=\{x\}$ if $x\in V$, and $P(Z)$ is closed if $Z\subseteq V$. If $G$ is a covering of $V$, let $G_x=\{X\in G:x\in X\}$. If $X$ is a set and $Y$ is a set, let $|X|$ denote the cardinal number of $X$ and $X-Y=\{x\in X:x\notin Y\}$. Let $n$ be an integer such that $n\geqq -1$. $\dim_P V$ is defined as follows: $\dim_P V=-1$ if $V=\varnothing$. If $V\neq\varnothing$, then $\dim_P V=n$ if (1) for each finite open covering $G$ of $V$, there is an open refinement $H$ of $G$ such that $|H_x|\leqq n+1$ if $x\in V$; and (2) there is a finite open covering $G$ of $V$ such that if $H$ is an open refinement of $G$, then $|H_x|\geqq n+1$ for some $x\in V$. We say that $P$ *has property* $(*)$ if for each nonempty open $Y\subseteq V$ and each $X\subseteq V$ such that $P(X)\neq V$ and $x\notin P(X-\{x\})$ whenever $x\in X$ and each $x\in[V-P(X)]$, $[Y-P(X)]\cap P(X\cup\{x\})\neq\varnothing$. THEOREM. *If $V$ is a metric space, $P$ has property* $(*)$, *$B\subseteq V$, $B$ is finite, $P(B)=V$ and $x\notin P(B-\{x\})$ if $x\in B$, then* $\dim_P V=|B|-1$.

1. **Introduction.** It is known [**5**, pp. 9, 93–99] that the covering dimension of each finite-dimensional Euclidean space $E^n$ is $n$, the usual dimension. The purpose of this paper is to present a short proof of this simply stated fact.

It is crucial that each finite-dimensional Euclidean space is a topological space $V$ in which there is *a structure* $P:2^V\to 2^V$ [**4**, p. 317] such that $P$ is *a closure structure* having *the exchange property* ([**2**], [**3**], and [**4**]), $P(\varnothing)=\varnothing$, $P(\{x\})=\{x\}$ for each $x\in V$, and $P(Z)$ is closed for each $Z\subseteq V$. Indeed, if $V=E^n$, then *the linear variety structure in* $V$ will suffice as $P$, that is, if $X\subseteq V$, then $P(X)$ is the collection of all finite linear combinations of elements of $X$ with coefficients summing to 1.

Consider a structure $P$ in a set $V$ and a subset $X$ of $V$. By definition, $X$ is *P-independent* ([**2**] and [**3**]) if $x\notin P(X-\{x\})$ for each $x\in X$; $X$ is *a P-basis of* $V$ if $X$ is $P$-independent and $P(X)=V$. By definition, *the P-dimension of* $V$, $P$-dim $V$, exists if any two $P$-bases of $V$ have the same cardinal number. If $P$-dim $V$ exists, then $P$-dim $V$ is the cardinal number

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of a $P$-basis of $V$. It is known ([2] and [3]) that if $P$ is a closure structure having the exchange property and $V$ has a finite $P$-basis, then $P$-dim $V$ exists.

If $G$ is a covering of a set $V$ and $x \in V$, then the symbol $G_x$ shall denote $\{X \in G : x \in X\}$. If $X$ is a set and $Y$ is a set, the symbol $X-Y$ shall denote $\{x \in X : x \notin Y\}$, and the symbol $|X|$ shall denote the cardinal number of $X$. Throughout the remainder of this paper it is assumed that $V$ is a topological space and $P$ is a structure in $V$ such that $P(\varnothing)=\varnothing$, $P(\{x\})=\{x\}$ for each $x \in V$ and $P(Z)$ is closed for each $Z \subseteq V$.

*The covering dimension of $V$ relative to $P$*, $\dim_P V$, is defined as follows: $\dim_P V=-1$ if $V=\varnothing$. If $V \neq \varnothing$ and $n$ is a cardinal number, then $\dim_P V = n$ if (1) and (2) are true: (1) For each finite open covering $G$ of $V$, there is an open refinement $H$ of $G$ [an open covering of $V$ such that if $X \in H$, then $X \subseteq Y$ for some $Y \in G$] such that $|H_x| \leqq n+1$ for each $x \in V$, and (2) There is a finite open covering $G$ of $V$ such that if $H$ is an open refinement of $G$, then $|H_x| \geqq n+1$ for some $x \in V$.

We say that *$P$ has property* $(*)$ if for each nonempty open subset $Y$ of $V$ and each $P$-independent subset $X$ of $V$ such that $X$ is not a $P$-basis of $V$ and each $x \in [V-P(X)]$, $Y-P(X)$ contains an element of $P(X \cup \{x\})$.

It is shown (Theorem 1) that if $G$ is a finite open covering of $V$ and $B$ is a $P$-basis of $V$, then there is an open refinement $H_B$ of $G$ such that $|(H_B)_x| \leqq |B|$ for each $x \in V$; and (Theorem 2) that if $V$ is a metric space and $P$ has property $(*)$ while $B$ is a finite $P$-basis of $V$, then there is a finite open covering $G_B$ of $V$ such that if $H$ is an open refinement of $G_B$, then $|H_x| \geqq |B|$ for some $x \in V$. It follows (Theorem 3) that if $V$ is a metric space and $P$ has property $(*)$ while $B$ is a finite $P$-basis of $V$, then $\dim_P V = |B|-1$.

2. **Main results.** If $V$ is a metric space, then the following notation will be used: If $r$ is a positive real number and $X$ is a nonempty subset of $V$, then the symbol $X_r$ shall denote $\{x \in V : d(x, X) < r\}$, where $d$ is the metric on $V$. The term "poset" [1, p. 1] will be used to refer to a pair $(W, R)$ such that $W$ is a set and $R$ is a partial order relation on $W$.

THEOREM 1. *If $G$ is a finite open covering of $V$ and $B$ is a $P$-basis of $V$, then there is an open refinement $H_B$ of $G$ such that $|(H_B)_x| \leqq |B|$ for each $x \in V$.*

PROOF. Assume that $G$ is a finite open covering of $V$, and that $B$ is a $P$-basis of $V$. Since $P(\varnothing)=\varnothing$ and $P(\{x\})=\{x\}$ for each $x \in V$, it follows that if $B=\varnothing$ or $B=\{x\}$ for some $x \in V$, then $V=P(B)=B$, so that $G=\{V\}$. Hence, if $B=\varnothing$ or $B=\{x\}$ for some $x \in V$, then let $H_B=G$. Consider the case that $|B|>1$. Let $b \in B$. Using Hausdorff's maximal principle [1,

p. 192], extend the chain $\{\varnothing\}$ of the poset $(2^{B-\{b\}}, \subseteq)$ to a maximal chain $K$ of $(2^{B-\{b\}}, \subseteq)$. Since $G$ is finite, then $\bigcap G_x$ is open for each $x \in V$ and each element of the poset $(\{G_x : x \in V\}, \subseteq) = \{G_x : x \in V\}$ is preceded by some minimal element of $\{G_x : x \in V\}$. Since $P$ is monotone and $B$ is $P$-independent, it follows that each subset of $B$ is $P$-independent. Hence, since $\varnothing \in K$ and $P(\varnothing) = \varnothing$, it follows that the collection $H$ of all $(\bigcap G_x) \cap [V-P(X)]$ such that $G_x$ is a minimal element of $\{G_y : y \in V\}$ and $X \in K$ is an open refinement of $G$. Assume that $x \in V$. Choose a minimal element $G_c$ of $\{G_y : y \in V\}$ such that $G_c \subseteq G_x$. Then $\bigcap G_x \subseteq \bigcap G_c$ while $x \in \bigcap G_x$. It follows that the elements of $H_x$ are among the sets $(\bigcap G_y) \cap [V-P(X)]$ such that $X \in K$ and $G_y$ is a minimal element of $\{G_z : z \in V\}$ while $|\{V-P(X) : X \in K\}| = |B|$. Therefore, $|H_x| \leqq |B|$. Let $H_B = H$. The proof is complete.

THEOREM 2. *If $V$ is a metric space and $P$ has property $(*)$ while $B$ is a finite $P$-basis of $V$, then there is a finite open covering $G_B$ of $V$ such that if $H$ is an open refinement of $G_B$, then $|H_x| \geqq |B|$ for some $x \in V$.*

PROOF. Assume that $V$ is a metric space such that $P$ has property $(*)$ while $B$ is a finite $P$-basis of $V$. If $B = \varnothing$ or $B = \{x\}$ for some $x \in V$, then let $G_B = \{V\}$. Consider the case that $|B| > 1$. Let $n$ be a positive integer such that $|B| = n+1$. Let $B$ consist of exactly $n+1$ elements $x_i$ of $V$, with $1 \leqq i \leqq n+1$. Let $X_{n+1} = B$. If $0 \leqq k \leqq n$, then let $X_{n-k} = X_{(n-k)+1} - \{x_{n-k}\}$. Let $r$ be a positive real number. Let $G$ consist of precisely the following sets: $P(X_k)_r$, with $1 \leqq k \leqq n$ and $X_0 = \varnothing$. Since $P(Z)$ is closed for each $Z \subseteq V$, it follows that $G$ is a finite collection of open subsets of $V$. Consider any element $x$ of $V$. If $x \in P(X_1)$, then $x \in [P(X_1)_r - P(X_{k-1})] \subseteq \bigcup G$. If $x \notin P(X_1)$, let $m$ be the largest positive integer such that $x \in P(X_m)$, so that $x \in [P(X_{m+1})_r - P(X_m)] \subseteq \bigcup G$. It follows that $G$ is a finite open covering of $V$. Assume that $H$ is an open refinement of $G$. If $1 \leqq k \leqq n+1$, let $H_k$ be the collection of all $X \in H$ such that $x \in X$ for some $x \in [P(X_k) - P(X_{k-1})]$. Since $x_1 \in [P(X_1) - P(X_0)]$, then $H_1$ contains an element $Y_1$. It follows from $(*)$ that $Y_1 - P(X_1)$ contains an element $y_1$ of $P(X_2)$, so that $y_1 \in [P(X_2) - P(X_1)]$. Hence $y_1 \in Y_2$ for some $Y_2 \in H_2$ such that $Y_2 \notin H_1$. If $n \geqq 2$, then it follows from $(*)$ that $(Y_1 \cap Y_2) - P(X_2)$ contains an element $y_2$ of $P(X_3)$, so that $y_2 \in [P(X_3) - P(X_2)]$ and $Y_2 \notin [P(X_i) - P(X_{i-1})]$ if $1 \leqq i \leqq 2$. It follows by induction that there is a collection $\{Y_i : 1 \leqq i \leqq n\}$ of elements of $H$ such that if $1 \leqq k \leqq n$, then $(\bigcap \{Y_i : 1 \leqq i \leqq k\}) - P(X_k)$ contains an element $y_k$ of $P(X_{k+1})$ while $Y_i \notin H_j$ if $1 \leqq j < i \leqq k$. It follows that $y_n \in [P(X_{n+1}) - P(X_n)]$ and $y_n \notin [P(X_i) - P(X_{i-1})]$ if $1 \leqq i \leqq n$. Let $x = y_n$. Then $x \in Y_{n+1}$ for some $Y_{n+1} \in H_{n+1}$ such that $Y_{n+1} \notin H_i$ if $1 \leqq i \leqq n$. It follows that $\{Y_i : 1 \leqq i \leqq n+1\} \subseteq H_x$, so that $|H_x| \geqq n+1$. Therefore, $|H_y| \geqq |B|$ for some $y \in V$. The proof is complete.

THEOREM 3. *If $V$ is a metric space and $P$ has property* $(*)$ *while $B$ is a finite $P$-basis of $V$, then* $\dim_P V=|B|-1$.

PROOF. Suppose that $V$ is a metric space such that $P$ has property $(*)$ while $B$ is a finite $P$-basis of $V$. It follows from Theorem 1 that if $G$ is a finite open covering of $V$, then there is an open refinement $H_B$ of $G$ such that $|(H_B)_x|\leqq|B|$ for each $x\in V$. Application of Theorem 2 yields a finite open covering $G_B$ of $V$ such that if $H$ is an open refinement of $G_B$, then $|H_x|\geqq|B|$ for some $x\in V$. Therefore, $\dim_P V=|B|-1$. The proof is complete.

COROLLARY. *If $V$ is a metric space and $P$ has property* $(*)$ *while $V$ has a finite $P$-basis and* $P$-dim $V$ *exists, then* $\dim_P V=[P\text{-dim } V]-1$.

The linear variety structure $Q$ in $E^n$ is a closure structure having the exchange property and property $(*)$, $Q$-dim $E^n$ exists and $E^n$ has a finite $Q$-basis of exactly $n+1$ elements. Therefore, $\dim_Q E^n=n$.

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